Chapter I: ELECTROSTATICS

Translated from DR.Dilmi Mourad's cours

I.1 Introduction.

Electrostatics is the study of interactions between electric charges held fixed in a given reference frame (**stationary**).

To evaluate the interactions between charges, it is very convenient to introduce the concept of a field. In this chapter, we will introduce and study the electrostatic field (often referred to as the electric field) and the methods to calculate it when possible.

Quantification of electric charge:

It is an experimental fact that all isolable electric charges are integer multiples of a fundamental charge, equal to the charge of the proton. It seems that all protons have rigorously the same electric charge, denoted as $q_p = 1{,}602.10^{-19}\,C$ (C: coulomb). All electrons have a charge equal to $-q_p$.

The existence of positive and negative charges corresponds to the fact that the force between two stationary charges (which we will soon investigate) can be attractive or repulsive.

The gravitational force, on the other hand, is always attractive, which is why all masses are positive.

Nevertheless, the electrostatic laws that we will develop are independent of the quantification of charge. The forces will be of electrostatic nature. It is also called Coulombic force.

To facilitate the understanding of certain parts, throughout the course, we will always assume that charges are positive (unless indicated otherwise).

Electrification Phenomenon:

All objects become electrified, and there are several means to do so:

By friction.

By contact with an already electrified object.

By connecting the object to a terminal of an electrical generator.

Experiments show that objects can be schematically classified into two categories:

Conducting Materials, Insulating Materials

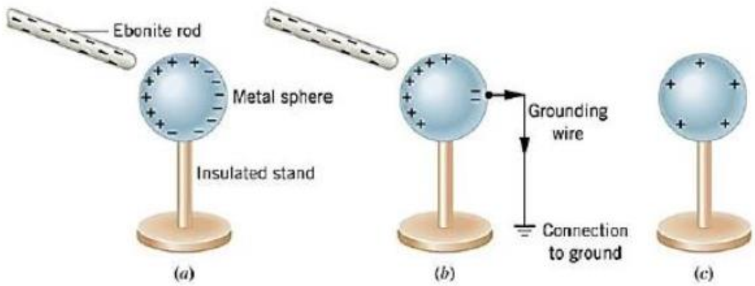
From an electrical perspective, we distinguish two main families of materials: insulators and conductors.

Insulating Materials (Dielectrics)

Bodies whose internal structure does not allow the passage of free electrons from one atom to another are called electrical insulators (such as ebonite, glass, porcelain, plastics, etc.). Insulators become charged by friction.

Conducting Materials:

In this type of material, charges are free to move and distribute themselves throughout the material due to electrostatic repulsion. A conductor becomes charged by induction.



At the macroscopic scale:

A negatively charged object has an excess of negative charges.

A positively charged object has a deficit of negative charges.

Charge Distributions:

The study of the physical properties of electrically charged bodies requires a mathematical description of the distribution of charges. We distinguish between discrete distributions and continuous distributions:

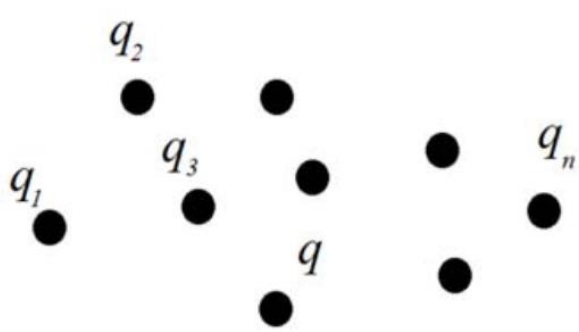
a) Discrete Distributions:

1- Point Charge:

A charge is considered as a point charge when the distance between the charge and the observer is much larger than the characteristic size of the charge.

2 - Discrete Distribution:

A set of charges discernible by an observer $q_1, q_2 \dots, q_n$

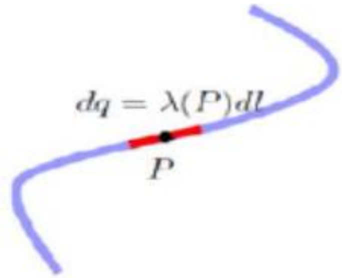


b) Continuous Distributions:

Assuming a macroscopic charge allowing the definition of an infinitesimal charge dq, to which the established formulas for a point charge can be applied before integrating over the distribution. This way, we define densities:

Linear Distribution:

The linear charge density $\boldsymbol{\lambda}$ is defined by:



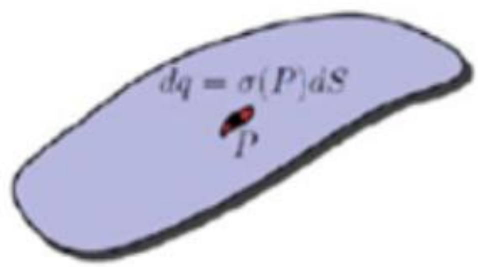
$$\boldsymbol{\lambda} = \lim_{\Delta l \to 0} \frac{\Delta q}{\Delta l} = \frac{dq}{dl}$$

$$\boldsymbol{\lambda} = \frac{dq}{dl} \quad [C.m^{-1}]$$

$$dq = \lambda\, dl$$

Surface Distribution:

The surface charge density $\boldsymbol{\sigma}$ is defined by:



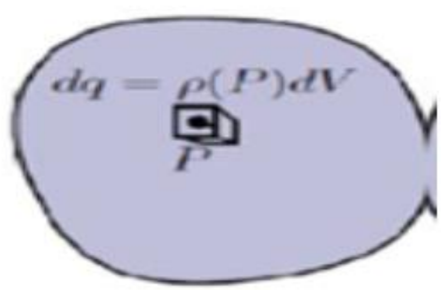
$$\boldsymbol{\sigma} = \lim_{\Delta s \to 0} \frac{\Delta q}{\Delta s} = \frac{dq}{ds}$$

$$\boldsymbol{\sigma} = \frac{dq}{ds} \quad [C.m^{-2}]$$

$$dq = \sigma\, ds$$

Volume Distribution:

The volume charge density $\boldsymbol{\rho}$ is defined as:



$$\boldsymbol{\rho} = \lim_{\Delta v \to 0} \frac{\Delta q}{\Delta v} = \frac{dq}{dv}$$

$$\boldsymbol{\rho} = \frac{dq}{dv} \quad [C.m^{-3}]$$

$$dq = \rho\, dv$$

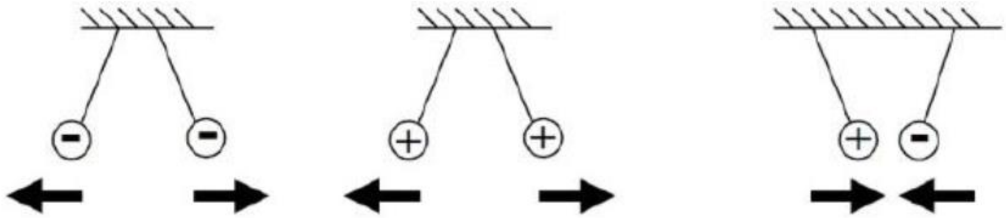
Coulomb Interactions (Electrostatic):

These are the reciprocal interactions exerted by two electrically charged systems on each other.

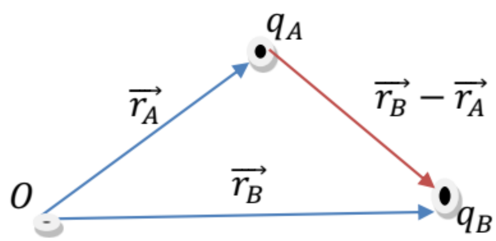
Experimentally, it is observed that:

Charges of the same sign **(+) (+)** or **(–) (–)** repel each other.

Charges of opposite signs **(+) (–)** or **(–) (+)** attract each other.



In the case of two point charges q_A and q_B,



Coulomb's Law

The properties of the electrostatic force exerted by one point charge q_1 on another point charge q_2 were determined by *Charles Auguste de Coulomb (1736-1806)* as follows:

1. The electrostatic force is directed along the line joining the two charges and
2. It is proportional to the product of the charges: it is attractive if the charges have opposite signs, and it is repulsive if the charges have the same sign.
3. It is inversely proportional to the square of the distance between the two charges.

The Coulomb's law expressing these properties is given by the following expression:

$$\overrightarrow{F_{AB}} = K \frac{q_A q_B}{|\vec{r_B} - \vec{r_A}|^2} \frac{\vec{r_B} - \vec{r_A}}{|\vec{r_B} - \vec{r_A}|}$$

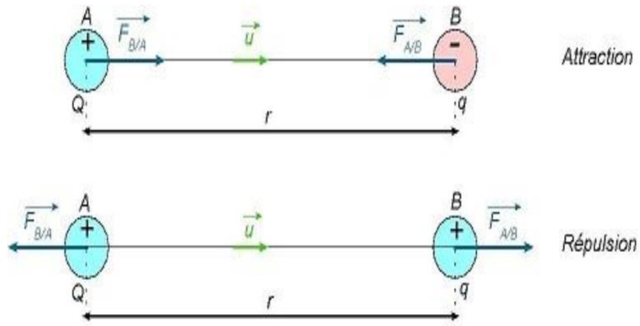
In the International System of Units (SI): $k = \frac{1}{4\pi\varepsilon_0} = 9.10^9\ S.I$ $\quad \overrightarrow{u_{AB}} = \frac{\vec{r_B}-\vec{r_A}}{|\vec{r_B}-\vec{r_A}|}$

With ε_0: vacuum permittivity. : $\varepsilon_0 = 8.854\ 10^{-12}\ F\ m^{-1} \Rightarrow K = 8.9877\ 10^9 \cong 9\ 10^9$

Alternatively: $\overrightarrow{F_{AB}} = K \frac{q_A q_B}{|\vec{r_B}-\vec{r_A}|^2} \overrightarrow{u_{AB}}$

Force exerted by on $\overrightarrow{F_{BA}} = K \frac{q_B q_A}{|\vec{r_A}-\vec{r_B}|^2} \overrightarrow{u_{BA}}$

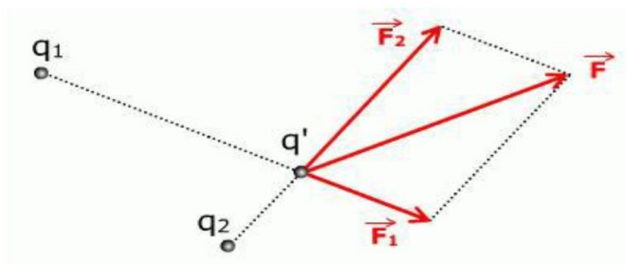
- 1st Case: $q_A\, q_B > 0$ Charges of the same sign (+) (+) or (−) (−).
- 2nd Case: $q_A\, q_B < 0$ Opposite charges (+) (−) or (−) (+).



Superposition principle

According to the additivity property of electrostatic forces acting on a charge q in the presence of two charges q_1 and q_2, the resultant of the forces is calculated as follows:

$$\vec{F} = \overrightarrow{F_1} + \overrightarrow{F_2} = K\frac{q_1 q}{{r_1}^2}\overrightarrow{u_1} + K\frac{q_2 q}{{r_2}^2}\overrightarrow{u_2}$$



$$\overrightarrow{F_{tot}} = \sum_{1}^{n} \overrightarrow{F_i}$$

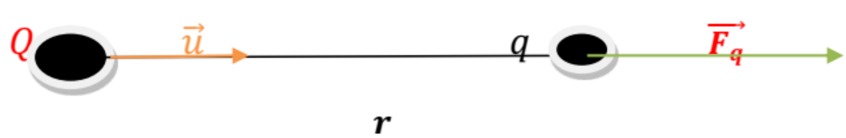
Electric Field:

In electrostatics:

An electric field is defined as a region of space where, at every point, a stationary charge q is subject to the action of an electric force. We then introduce a vector quantity $\vec{E}$ such that:

$$\vec{E} = \frac{Kq}{r^2}\ \overrightarrow{u_r}$$

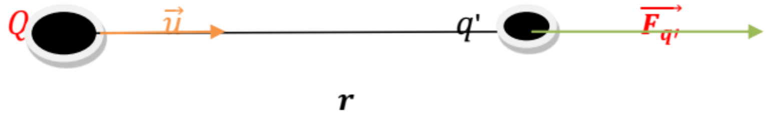
Consider a fixed charge Q and bring a charge q near a point M located at a distance r from it.



The Coulomb's law indicates that a charge q undergoes an electrostatic force.

$$\overrightarrow{F_q} = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r^2}\vec{u}$$

If we replace **$\grave{q}$** with **q**, the force becomes:



$$\overrightarrow{F_{q'}} = \frac{1}{4\pi\varepsilon_0}\frac{Qq'}{r^2}\vec{u}$$

The vector quantity **$\vec{F}/q$** is independent of **q** or **$\grave{q}$**:

$$\frac{\overrightarrow{F_q}}{q} = \frac{\overrightarrow{F_{q'}}}{q'} = \frac{1}{4\pi\varepsilon_0}\frac{Qq'}{r^2}\vec{u} = \vec{E} \iff \begin{cases} \overrightarrow{F_q} = q\vec{E} \\ \overrightarrow{F_{q'}} = q'\vec{E} \end{cases}$$

$\vec{E}$: depends only on the source charge **Q** and the distance from that source.

This quantity is defined as the electrostatic field created by the charge **Q**

$\vec{E}$: is a vector field defined at (almost) all points in space.

$$\boldsymbol{\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\overrightarrow{u_r}}$$

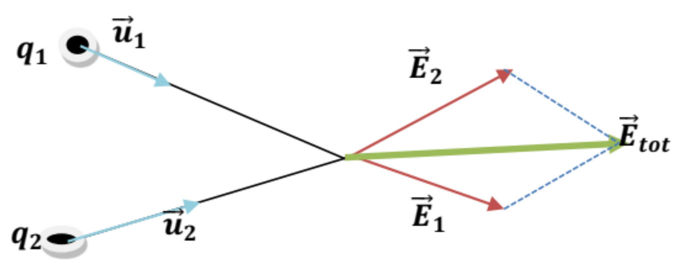
Electrostatic field created by two charges:

Now consider ***n*** charges q_i located at points P_i, what would be the electric field produced by this set of charges at point M?

As with forces, the principle of superposition also applies to electric fields. The total field $\vec{E}(M)$ is the vector sum of all contributions from each of the charges. Thus, we have:

$$\boldsymbol{\vec{E}_{tot} = \sum_{i=1}^{n}\vec{E}_i = \sum_{i=1}^{n}\frac{1}{4\pi\varepsilon_0}\frac{q_i}{{r_i}^2}\overrightarrow{u_i}}$$

$$\boldsymbol{\vec{E}_{tot} = \vec{E}_1 + \vec{E}_2 + \cdots\cdots + \vec{E}_n}$$



Composition of fields at a point

Electric fields created by continuous distributions:

In the case of continuous distributions, each portion of line, surface, or volume carrying charge dq creates an elementary field $d\vec{E}$. To obtain the total electric field at a point M, we need to sum (continuously) these elementary fields over the entire line, surface, or volume.

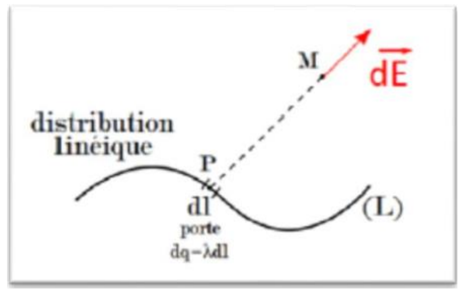
Thus, we resort to integrals:

Case of a line:

For a linear charge distribution characterized at each point on a curve Γ by the linear density:

$$\lambda = dq/dl$$

$$\vec{E} = \int \frac{dq}{4\pi\varepsilon_0}\frac{\vec{u}}{PM^2} = \int \frac{\lambda\, dl}{4\pi\varepsilon_0}\frac{\vec{u}}{PM^2}$$

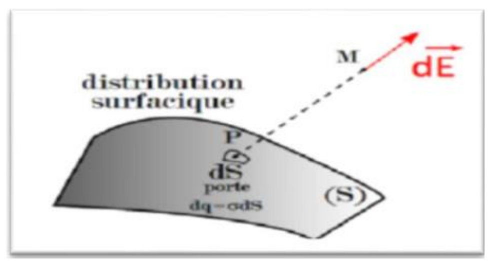


Case of a surface:

For a surface charge distribution characterized by the surface density $\sigma = dq/dS$ given at each point on a surface Σ, we will write similarly:

$$\vec{E} = \iint \frac{dq}{4\pi\varepsilon_0}\frac{\vec{u}}{PM^2} = \iint \frac{\sigma ds}{4\pi\varepsilon_0}\frac{\vec{u}}{PM^2}$$

$P \in (S)$



Case of volume:

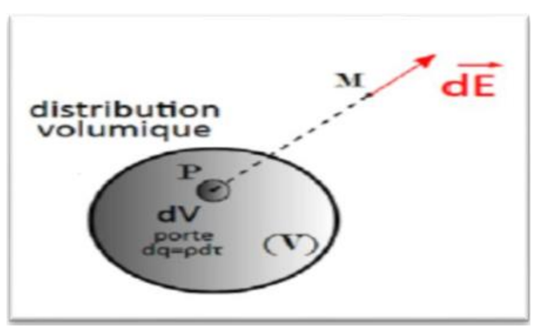
The distribution is characterized at each point P in the volume by the given volume charge density $\rho(P) = dq/dv$ where dq denotes the electric charge contained in the volume element dv surrounding the point P.

In the case where the charge distribution dq is uniform, we have:

$$\vec{E} = \iiint \frac{dq}{4\pi\varepsilon_0}\frac{\vec{u}}{PM^2} = \iiint \frac{\rho dv}{4\pi\varepsilon_0}\frac{\vec{u}}{PM^2}$$

$P \in (V)$

With $\vec{u} = \frac{\overrightarrow{PM}}{PM}$



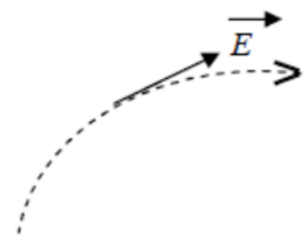
Field lines:

An electrostatic field line is a curve tangent at each point to the electrostatic field vector defined at that point. The set of field lines defines a mapping of the field.

Properties:

1. Two field lines never intersect at a point ***M*** unless the field ***E*** is zero at that point.
2. An electrostatic field line is not closed. It starts from a charge ***q*** and ends on a charge of opposite sign.
3. To determine the direction of the field at a point ***M*** on a field line, a positive charge should be placed there, and the direction and sense of the electrostatic force it experiences should be observed.

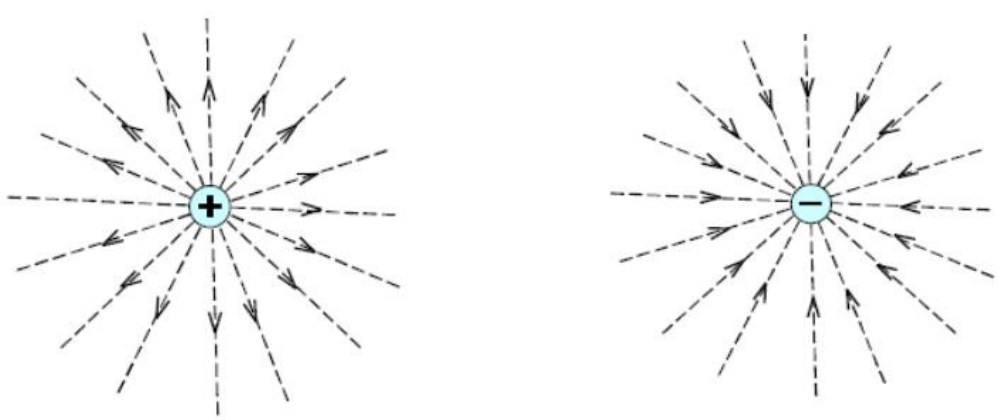
These direction and sense are the same as those of the field.



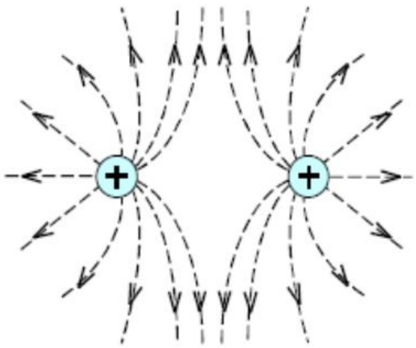
In the case of a point charge, the field lines are half-lines that intersect at the point where the charge is located. If the charge is positive, the field is directed outward; we say it is emanating. The same applies to the field lines. Conversely, for a negative charge, the field lines converge towards the charge; in this case, the field is directed towards the charge, representing the field lines due to a single source charge ***Q***.

If the source charge is positive **(+)**, the field is directed outward from the charge. If the charge is negative **(-)**, the field is directed inward towards the charge.

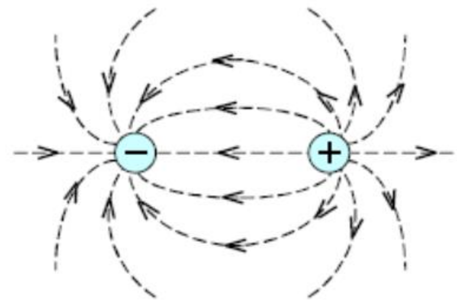
1- Point charges:



2- *Ensemble of two positive charges:*



3- *Opposite and distinct charges:*



Equipotential surface:

An equipotential surface is defined as a surface S where all points have the same potential ***V***.

a) The field lines are perpendicular to the equipotential surfaces they encounter.

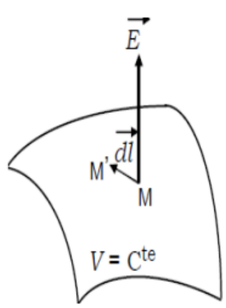
Considering a very small displacement on an equipotential surface, we find:

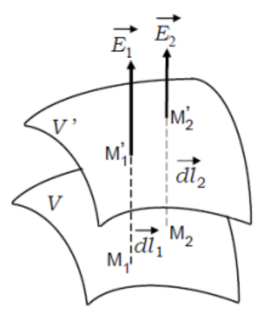
$$\overrightarrow{MM'}=\overrightarrow{dl}$$

$$dV = -\vec{E}.\overrightarrow{dl} = -\vec{E}.\overrightarrow{MM'} = 0$$

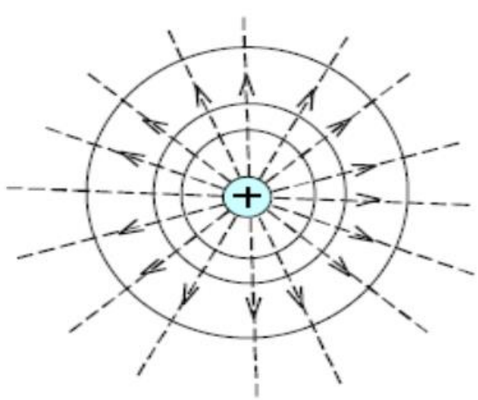
Therefore, $\vec{E}$ is perpendicular to $\overrightarrow{MM'}$.

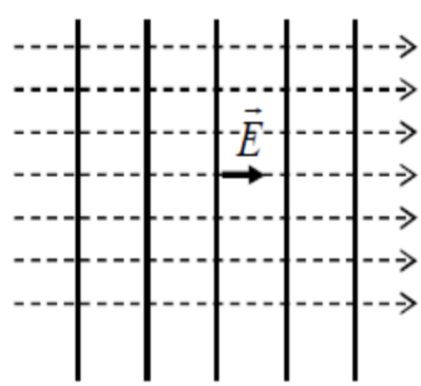
$$\vec{E} \perp \overrightarrow{MM'}$$

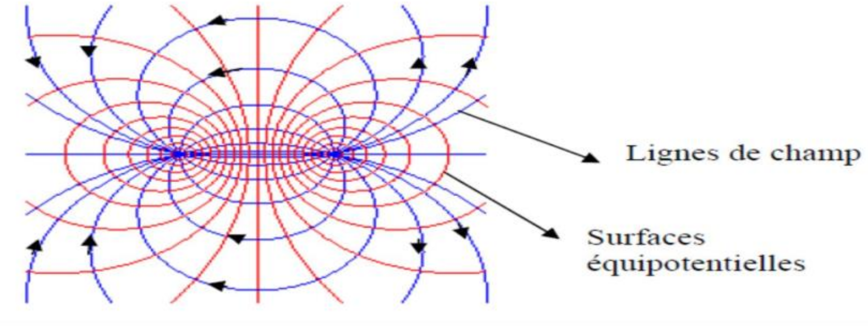




In the case of a uniform field, the field lines are straight and parallel. The equipotential surfaces are planes perpendicular to these lines.







Electrostatic potential:

Consider a charge $\boldsymbol{q}$ placed at the origin of a coordinate system. Let's calculate the circulation of $\vec{\boldsymbol{E}}$ corresponding to a small displacement $\overrightarrow{\boldsymbol{dE}}(\boldsymbol{M})$ from the point $\boldsymbol{M}$.

$$dC = \vec{E}.\overrightarrow{dM} = \frac{q}{4\pi\varepsilon_0 r^2}\vec{u}.\overrightarrow{dM} = \frac{q}{4\pi\varepsilon_0 r^2}dr = d\left(-\frac{q}{4\pi\varepsilon_0 r} + K\right)$$

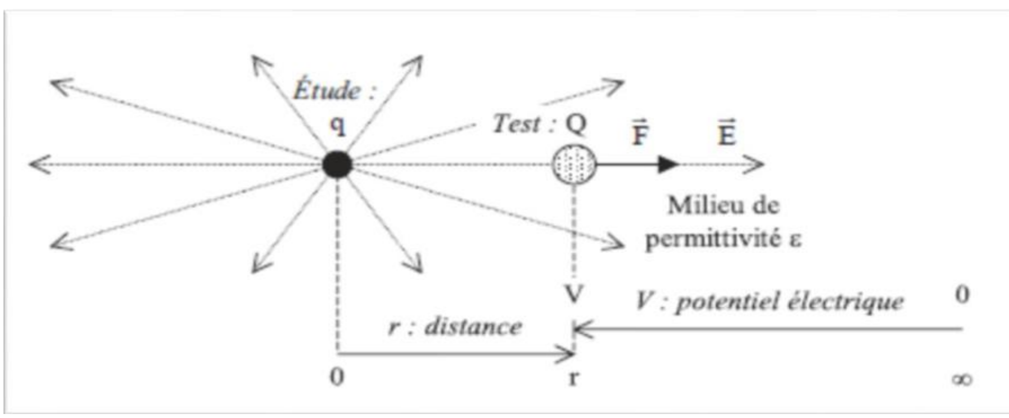
$$dC = -dV \ \ avec \ V = \frac{q}{4\pi\varepsilon_0 r} + K$$

For a single charge, the circulation of a fictitious point in the presence of the field $\vec{\boldsymbol{E}}$ is a total differential:

$$dC = \vec{E}.\overrightarrow{dM} = -dV \quad \text{Avec } dV = \overrightarrow{grad}V\,.\overrightarrow{dM}$$

This leads us to conclude that for a point charge:

$$\boldsymbol{\vec{E} = -\overrightarrow{grad}V \quad avec \ \ V = \frac{q}{4\pi\varepsilon_0 r} + K}$$



Electrostatic potential created by two charges:

$$\boldsymbol{V_{tot} = \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} \frac{1}{4\pi\varepsilon_0}\ \frac{q_i}{r_i} \ + K}$$

Electrostatic potential created by continuous distributions:

Linear charge distribution

$$V = \int \frac{dq}{4\pi\varepsilon_0 r} = \int \frac{\lambda dl}{4\pi\varepsilon_0 r} \qquad P \in (L)$$

Surface charge distribution

$$V = \iint \frac{dq}{4\pi\varepsilon_0 r} = \iint \frac{\sigma ds}{4\pi\varepsilon_0 r} \qquad P \in (S)$$

Volume charge distribution

$$V = \iiint \frac{dq}{4\pi\varepsilon_0 r} = \iiint \frac{\rho dv}{4\pi\varepsilon_0 r} \qquad P \in (V)$$