CALCULUS REMINDER

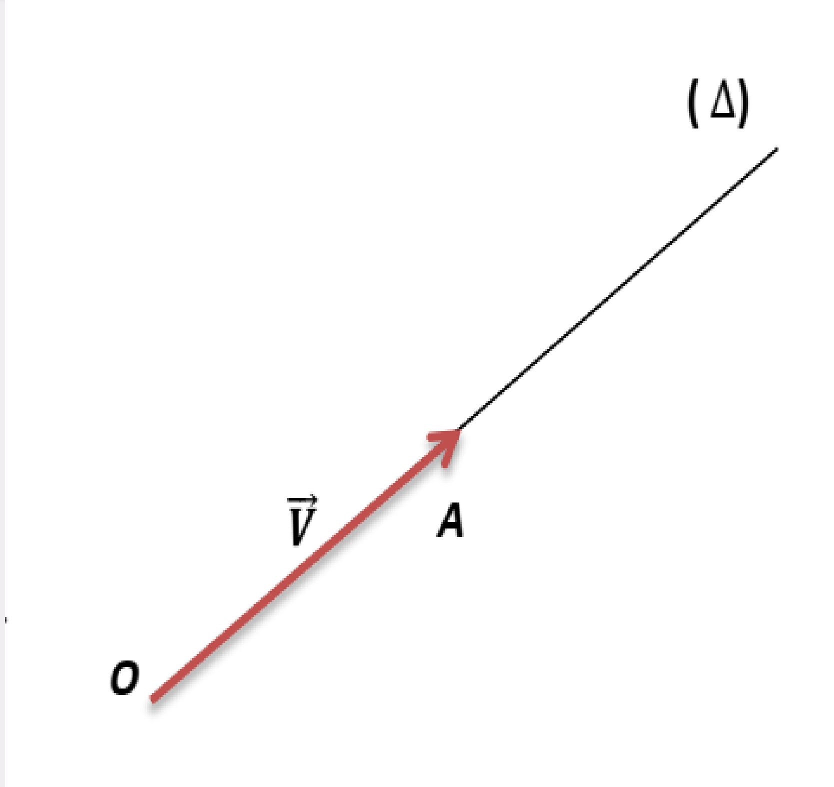
1. Concept of Vector

1.1. Definition: A vector is a mathematical entity that represents an element of a vector space $\mathbb{E}^3$ associated with an affine space (of points) $\mathbb{R}^3$, where a direction, a sense, a magnitude, and a point of application are defined

- «**O**» Point of application.
- «**Δ**» Direction (line of action).
- «$||\overrightarrow{\boldsymbol{OA}}||$» (in the orthonormal basis $(\vec{\boldsymbol{\imath}}, \vec{\boldsymbol{\jmath}}, \vec{\boldsymbol{k}})$.

$||\overrightarrow{\boldsymbol{OA}}|| = \sqrt{\boldsymbol{x^2 + y^2 + z^2}}$ is the magnitude of the vector.

- From **O** to **A** is the direction.

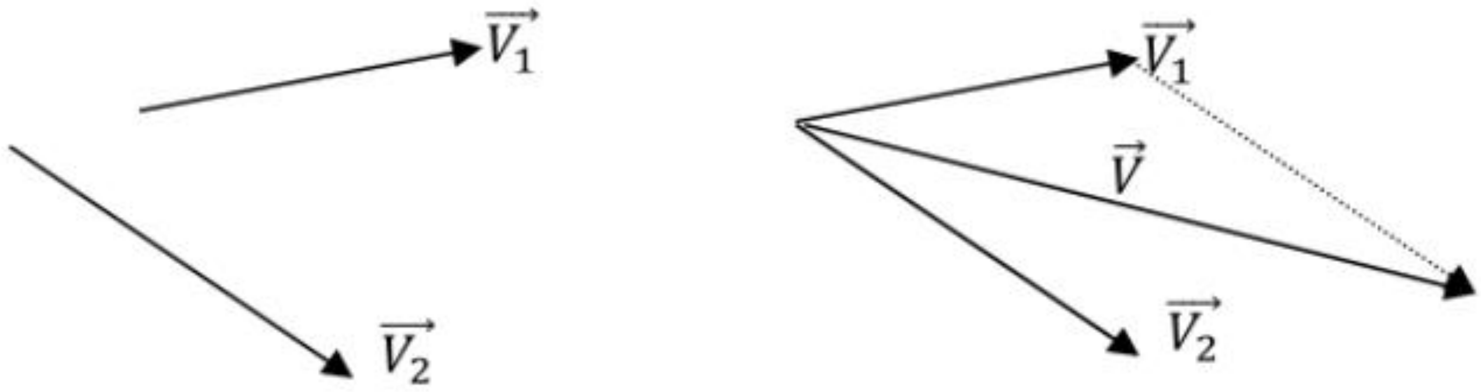


2. Vector Operations

2.1 Vector Addition: Relative to an orthonormal basis ($\vec{i}, \vec{j}, \vec{k}$), the sum of two vectors is a vector where the components add up pairwise.

$$\overrightarrow{V_1} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k} \quad \text{et} \quad \overrightarrow{V_2} = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}$$

$$\Rightarrow \quad \vec{V} = \overrightarrow{V_1} + \overrightarrow{V_2} = (x_1 + x_2)\vec{i} + (y_1 + y_2)\vec{j} + (z_1 + z_2)\vec{k}$$



2.2 Vector Subtraction:

The difference between two vectors $\vec{V}$ and $\vec{U}$, denoted as $\vec{V}$ - $\vec{U}$, is the sum of the first vector and the negation (opposite) of the second vector.

$$\vec{V} = \overrightarrow{V_1} - \overrightarrow{V_2} = \overrightarrow{V_1} + (-\overrightarrow{V_2})$$

2.3 Vector Products:

a) Dot Product and Projection: The dot product of two vectors $\vec{V}_1$ and $\vec{V}_2$, denoted as $\vec{V}_1 \cdot \vec{V}_2$, is a scalar represented as $\vec{V}_1 \cdot \vec{V}_2$. It is equal to the sum of the pairwise products of their components.

$$\overrightarrow{V_1} = x_1\vec{\imath} + y_1\vec{\jmath} + z_1\vec{k} \quad \text{et} \quad \overrightarrow{V_2} = x_2\vec{\imath} + y_2\vec{\jmath} + z_2\vec{k}$$

⇨ $=> \vec{V} = \overrightarrow{V_1}.\overrightarrow{V_2} = (x_1.x_2) + (y_1.y_2) + (z_1.z_2)$

Note: For the unit vectors of the orthonormal basis, we have

- ✓ $\vec{\imath}.\vec{\imath} = \vec{\jmath}.\vec{\jmath} = \vec{k}.\vec{k} = 1$
- ✓ $\vec{\imath}.\vec{\jmath} = \vec{\imath}.\vec{k} = \vec{\jmath}.\vec{k} = 0$

The square of the magnitude of the vector is:

$$\vec{V}.\vec{V} = (x.x) + (y.y) + (z.z) = x^2 + y^2 + z^2 = V^2$$

$$=> \|\vec{V}\| = V = \sqrt{x^2 + y^2 + z^2}$$

The dot product can also be defined in the following way:

$$\vec{V_1}.\vec{V_2} = \|\vec{V_1}\|.\|\vec{V_2}\|.\cos(\vec{V_1};\vec{V_2}) = \|\vec{V_1}\|.\|\vec{V_2}\|.\cos(\theta)$$

$$\vec{V_1}.\vec{V_1} = \|\vec{V_1}\|.\|\vec{V_1}\|.\cos(\vec{V_1};\vec{V_1}) = {V_1}^2$$

The dot product is zero if:

$$\|\vec{V_1}\| = 0 \text{ , } \|\vec{V_2}\| = 0$$

$$\vec{V_1} \perp \vec{V_2}$$

b) Cross Product

The cross product of two vectors $\vec{V}_1$ and $\vec{V}_2$ is a vector denoted as $\vec{V}_1$

$$\vec{V_1} \wedge \vec{V_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} \vec{i} - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} \vec{j} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \vec{k}$$

$$\vec{V_1} \wedge \vec{V_2} = (y_1 z_2 - y_2 z_1)\vec{i} - (x_1 z_2 - x_2 z_1)\vec{j} + (x_1 y_2 - x_2 y_1)\vec{k}$$

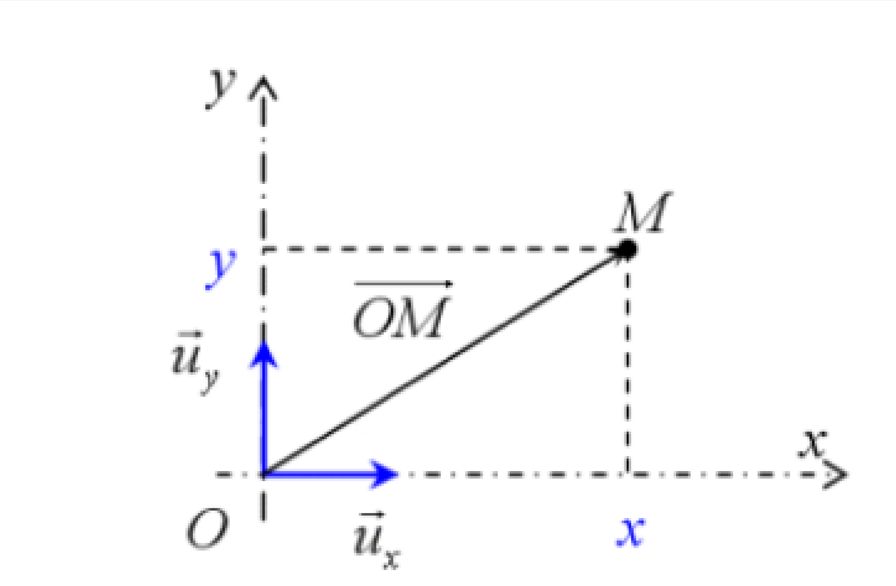
Coordinate System in the Plane

Cartesian coordinates: Any point in the plane can be located by its Cartesian coordinates in the orthonormal basis: $(\vec{i}, \vec{j})$. We can then write:

$$\overrightarrow{OM} = \vec{r} = x\vec{i} + y\vec{j}$$

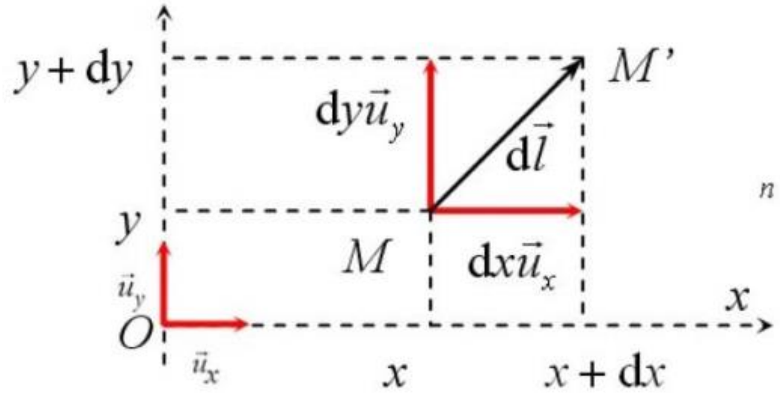
The magnitude is:

$$|\overrightarrow{OM}| = |\vec{r}| = \sqrt{x^2 + y^2}$$



Infinitesimal displacement: We consider the infinitesimal displacement of the point ***M (x,y)*** to the point ***M'(x+dx,y+dy)***. The displacement $\overrightarrow{\boldsymbol{M\grave{M}}}$ can then be written as:

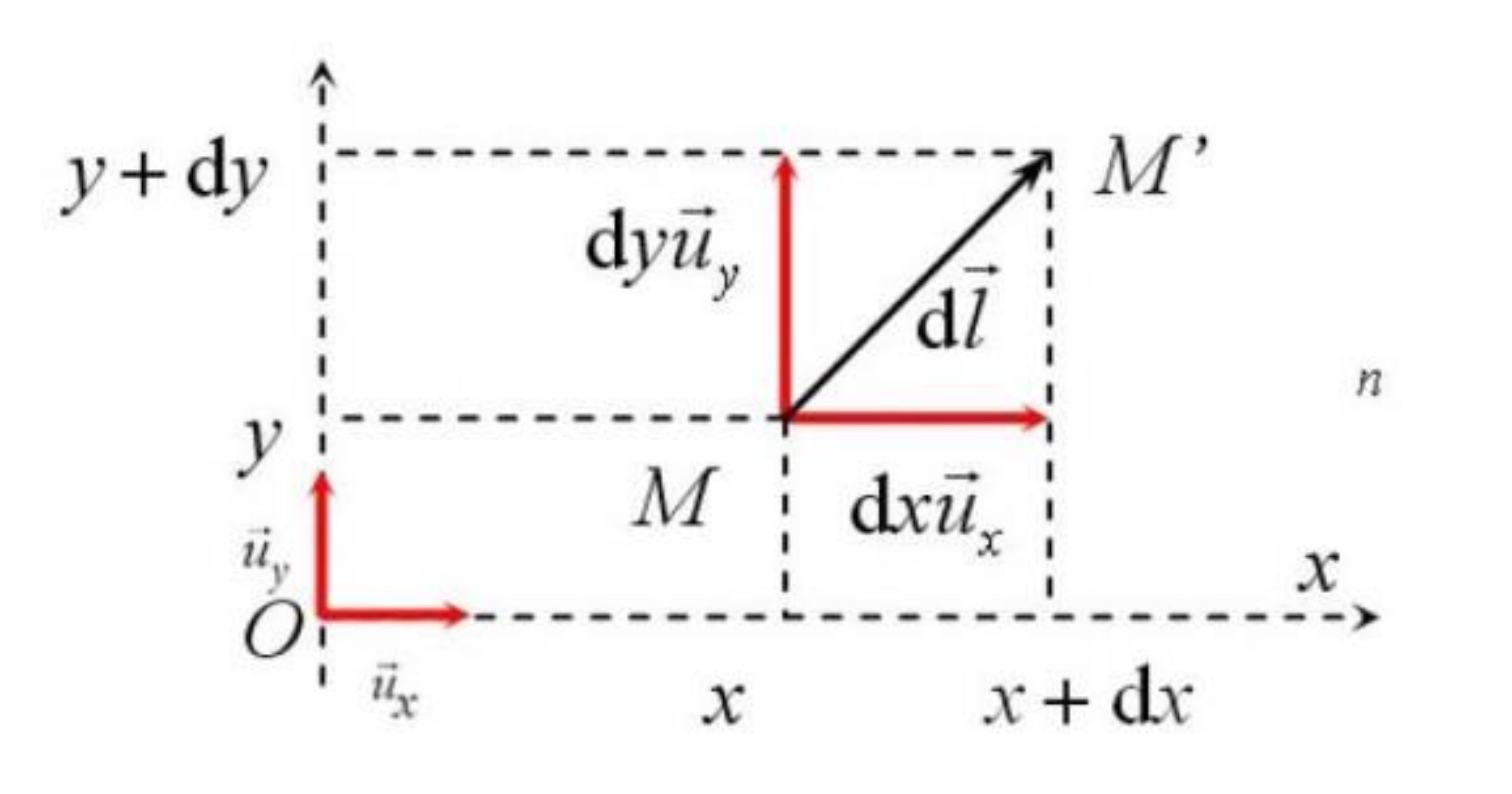
$$\overrightarrow{MM'} = dx\overrightarrow{u_x} + dy\overrightarrow{u_y}$$



Infinitesimal surface element:

We consider the infinitesimal surface generated by the displacement of the point ***M*** previously described. The area of this surface is given by

$$ds = dx.dy$$



$y + \mathrm{d}y$
M'
$\mathrm{d}y\vec{u}_y$
$\mathrm{d}\vec{l}$
n
y
M
$\mathrm{d}x\vec{u}_x$
x
$\vec{u}_y$
O
$\vec{u}_x$
x
$x + \mathrm{d}x$

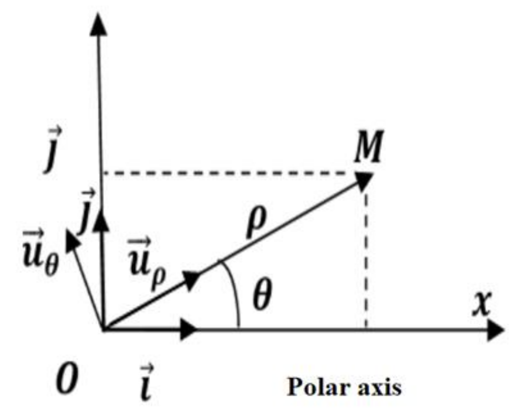
Polar coordinates: In the plane, a point is identified in Cartesian coordinates by its abscissa and ordinate (two distances). In polar coordinates, a point is identified by a distance and an angle defined as follows:

- $\boldsymbol{\rho}$: distance from the point $\boldsymbol{M}$ to the origin $\boldsymbol{O}$.
- $\boldsymbol{\theta}$: the angle in the counterclockwise (positive) direction, known as the polar angle.

$$\overrightarrow{OM} = \vec{r} = x\vec{\imath} + y\vec{\jmath} = \rho\overrightarrow{u_\rho} = \rho\cos\theta\,\vec{\imath} + \rho\sin\theta\,\vec{\jmath}$$

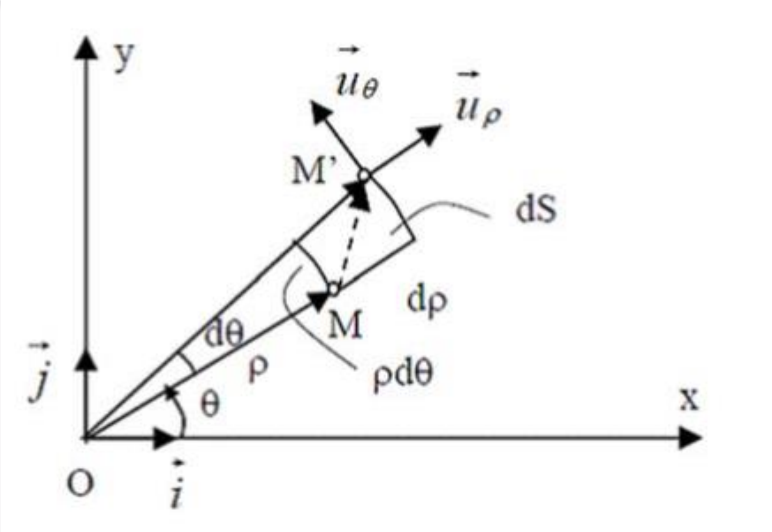
$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

$$\Leftrightarrow \qquad \begin{cases} \rho = \sqrt{x^2 + y^2} \\ \theta = arctg\,\dfrac{y}{x} \end{cases}$$



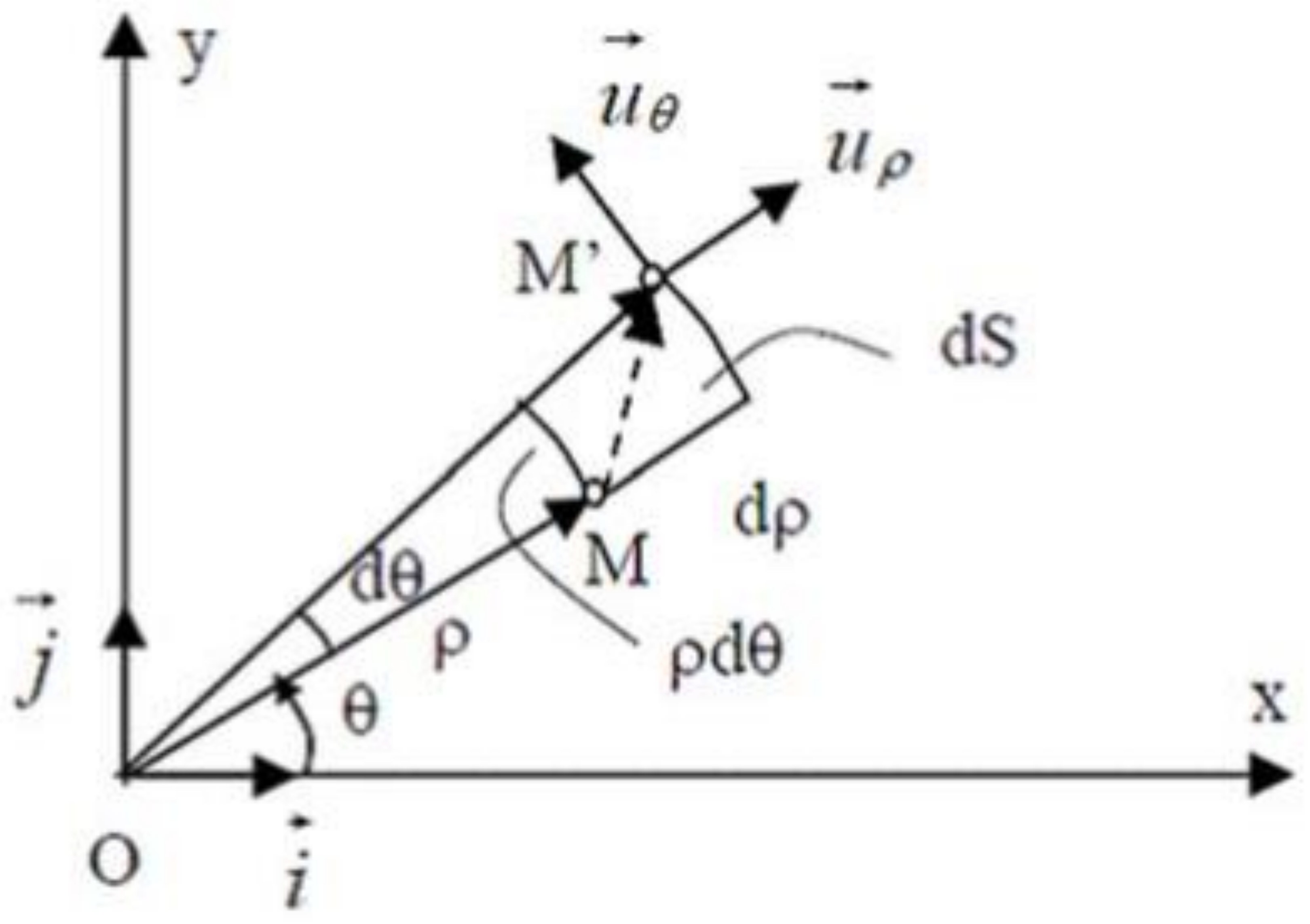
Infinitesimal displacement: We consider the infinitesimal displacement of point ***M(ρ,θ)*** to point ***M'(ρ+dρ,θ+dθ).*** The displacement $\overrightarrow{MM'}$ can then be written as:

$$\overrightarrow{MM'} = d\rho\overrightarrow{u_\rho} + \rho d\theta\overrightarrow{u_\theta}$$



Infinitesimal surface element: We consider the infinitesimal surface generated by the previously described point's displacement. The area of this surface is given by:

$$ds = \rho . d\rho . d\theta$$


y
$\vec{u}_\theta$
$\vec{u}_\rho$
M'
dS
dρ
M
dθ
ρ
ρdθ
$\vec{j}$
θ
x
O
$\vec{i}$

Coordinate System in the space:

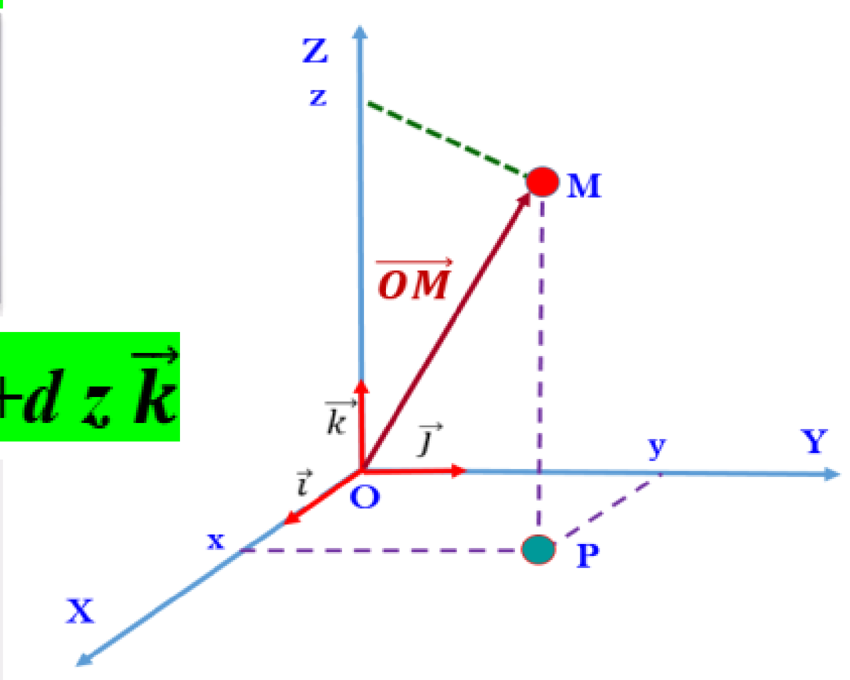
A) Cartesian coordinates:

A Cartesian coordinate system is defined by an origin point and three mutually perpendicular axes. The unit vectors along the axes are: $(\vec{i}, \vec{j}, \vec{k})$. Every point in space is identified by the three components of the vector $\vec{r}$.

$$\overrightarrow{OM} = \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

The displacement differential is easily found from its definition:

$$d\overrightarrow{OM} = d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

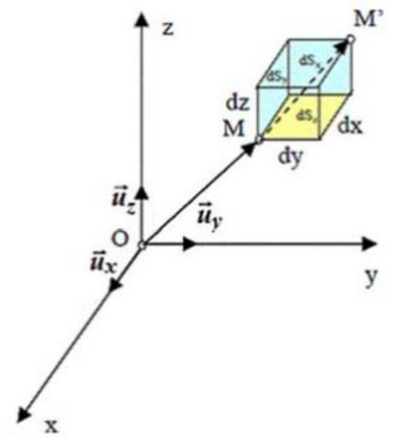


Infinitesimal surface element: Fixing one of the coordinates, the point moves within an elemental surface area:

$dS_x = dy.dz$ If we fix the abscissa ***x*;**

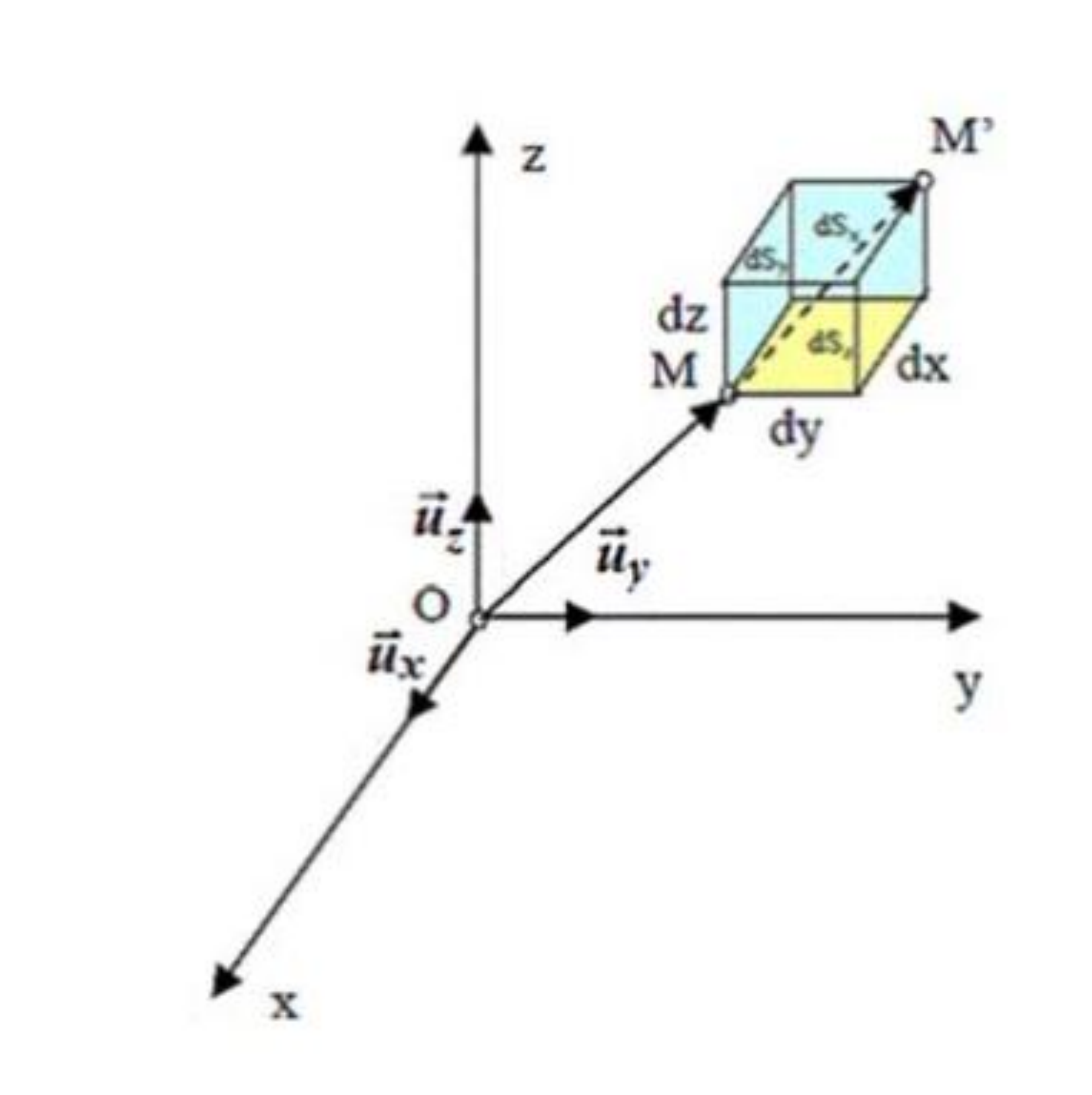
$dS_y = dx.dz$ If we fix the ordinate *y*;

$dS_z = dx.dy$ If we fix the height ***z*.**



Infinitesimal volume element: The displacement from $\boldsymbol{M}$ to $\boldsymbol{\grave{M}}$ generates an elementary volume bounded by six parallel surfaces, pairwise of which $\overrightarrow{\boldsymbol{M\grave{M}}}$ is a principal diagonal.

$$dV = dx.dy.dz$$

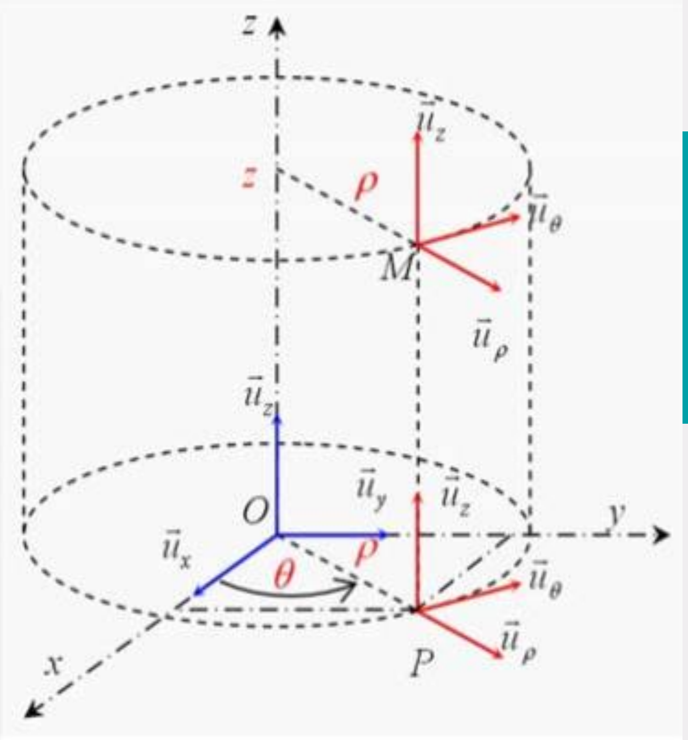


z
M'
dz
M
dx
dy
$\vec{u}_z$
$\vec{u}_y$
O
$\vec{u}_x$
y
x

Cylindrical coordinates: A point ***M*** in space can be identified by its cylindrical coordinates $\boldsymbol{\rho}, \boldsymbol{\theta}$ and $\boldsymbol{z}$ in the basis associated with the cylindrical coordinate system.

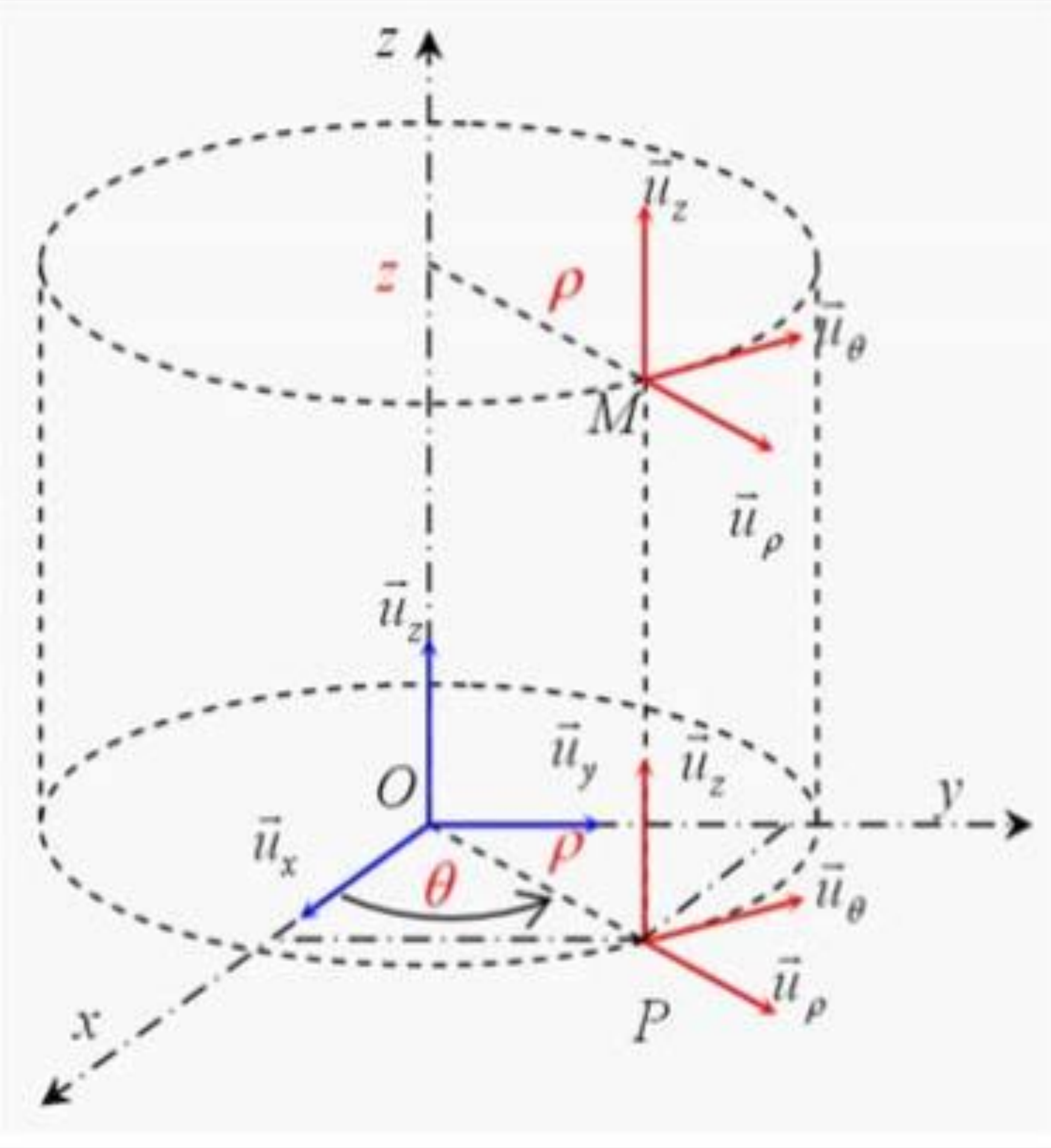
$$\overrightarrow{OM} = \vec{r} = \rho\overrightarrow{u_\rho} + z\vec{k}$$

$$|\overrightarrow{OM}| = |\vec{r}| = \sqrt{\rho^2 + z^2}$$



Infinitesimal displacement: We consider the infinitesimal displacement of point ***M(ρ,θ,z)*** to point ***M'(ρ+dρ, θ+dθ,z+dz)***. The displacement $\overrightarrow{MM'}$ can then be written as:

$$\overrightarrow{MM'} = d\rho\overrightarrow{u_\rho} + \rho d\theta\overrightarrow{u_\theta} + dz\overrightarrow{u_z}$$


z
$\vec{u}_z$
z
ρ
$\vec{u}_\theta$
M
$\vec{u}_\rho$
$\vec{u}_z$
$\vec{u}_y$
$\vec{u}_z$
O
y
$\vec{u}_x$
ρ
θ
$\vec{u}_\theta$
$\vec{u}_\rho$
P
x

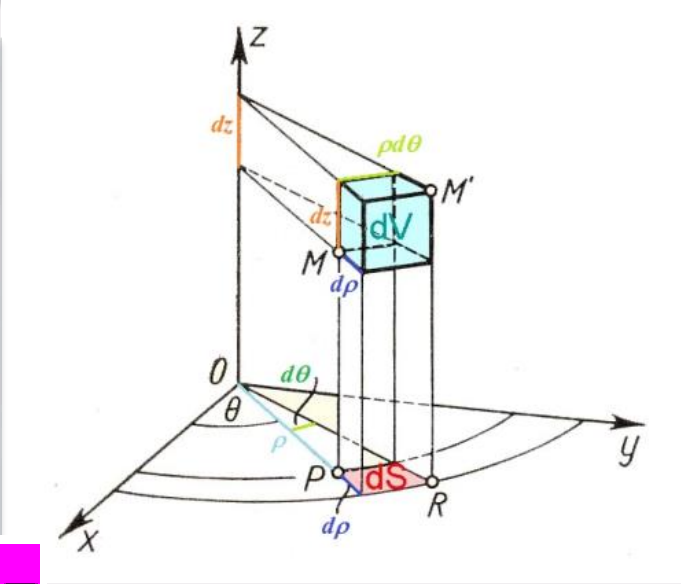
Infinitesimal surface element: Fixing one of the coordinates, the point ***M*** moves within an elementary surface area;

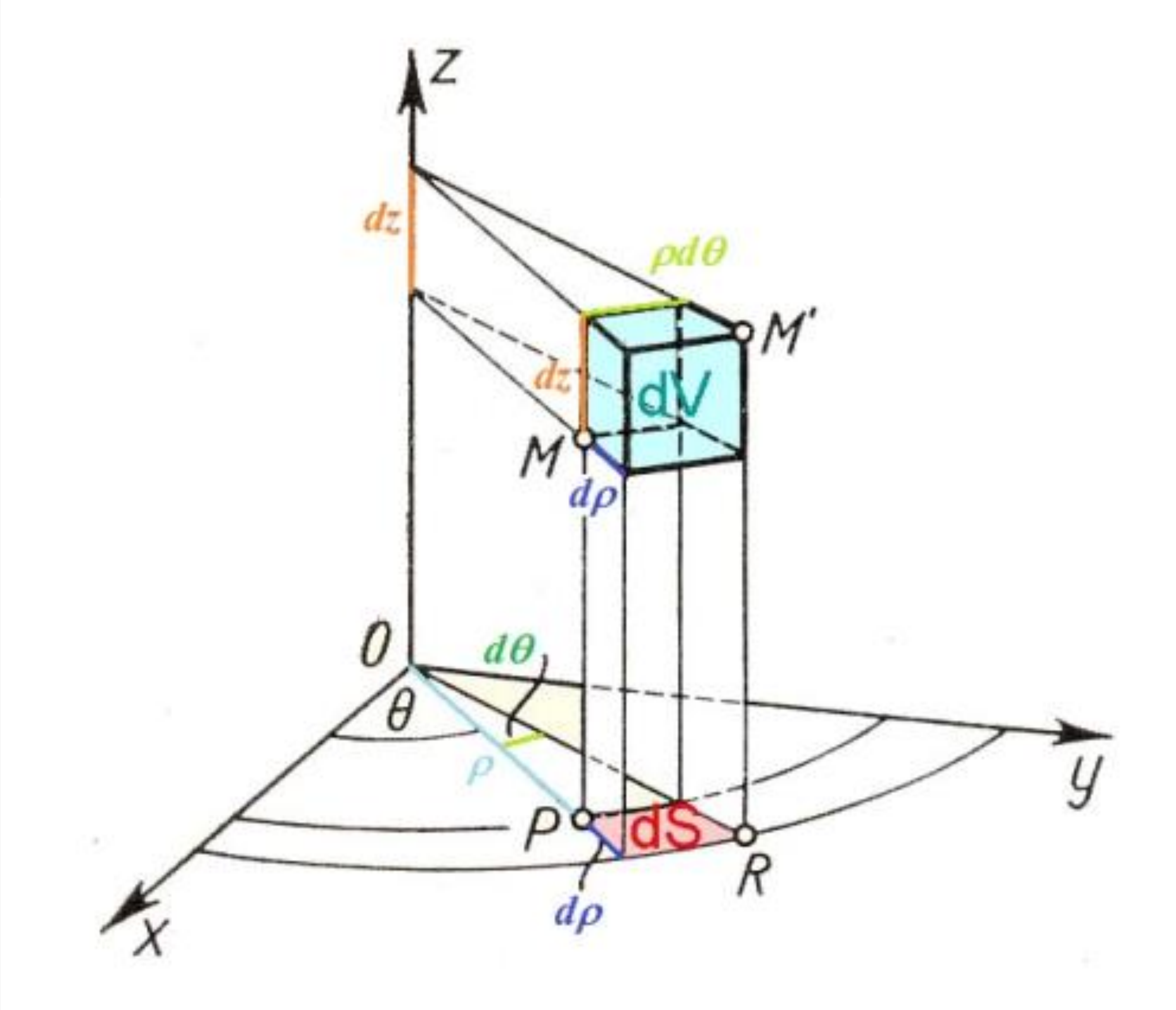
$dS_\rho = \rho d\theta . dz$	If we fix the radius $\boldsymbol{\rho}$.
$dS_\theta = d\rho . dz$	If we fix the angle $\boldsymbol{\theta}$.
$dS_z = d\rho . \rho . d\theta$	If we fix the height ***z***.

Infinitesimal volume element:

We consider the infinitesimal volume ***dV*** generated by the previously described point's ***M*** displacement. This volume is given by:

$$\boldsymbol{dV = \rho . d\rho . d\theta . dz}$$




z
dz
ρdθ
M'
dz
dV
M
dρ
O
dθ
θ
ρ
y
P
dS
R
dρ
x

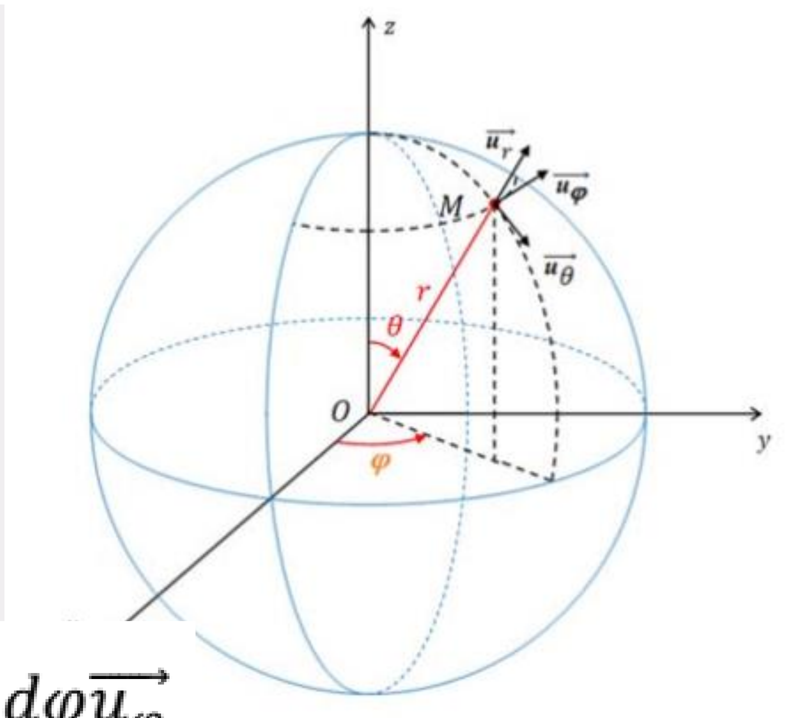
Spherical coordinates: A point *M* in space can be identified by its spherical coordinates $\boldsymbol{r},\boldsymbol{\theta}$, and $\boldsymbol{\varphi}$ in the basis associated with the spherical reference frame $(\vec{\boldsymbol{u}}_r, \vec{\boldsymbol{u}}_\theta, \vec{\boldsymbol{u}}_\varphi)$.

We can then write: $\overrightarrow{\mathbf{OM}} = r\vec{\mathbf{u}}_r = x\vec{\mathrm{i}} + y\,\vec{\mathrm{j}} + z\,\vec{\mathrm{k}}$

$$\overrightarrow{u_r} = \sin\theta\cos\varphi\,\vec{i} + \sin\theta\sin\varphi\,\vec{j} + \cos\theta\,\vec{k}$$

Infinitesimal displacement:

We consider the infinitesimal displacement of the point $\boldsymbol{M(\rho,\theta,\varphi)}$ to the point $\boldsymbol{M'(\rho+d\rho,\ \theta+d\theta,\ \varphi+d\varphi)}$. The displacement $\overrightarrow{\boldsymbol{MM'}}$ can then be written as:



$$\overrightarrow{MM'} = dr\overrightarrow{u_r} + r\,d\theta\overrightarrow{u_\theta} + r\sin\theta\,d\varphi\overrightarrow{u_\varphi}$$

Infinitesimal surface element:

$$dS_r = r^2 \sin\theta\, d\theta\, d\varphi$$

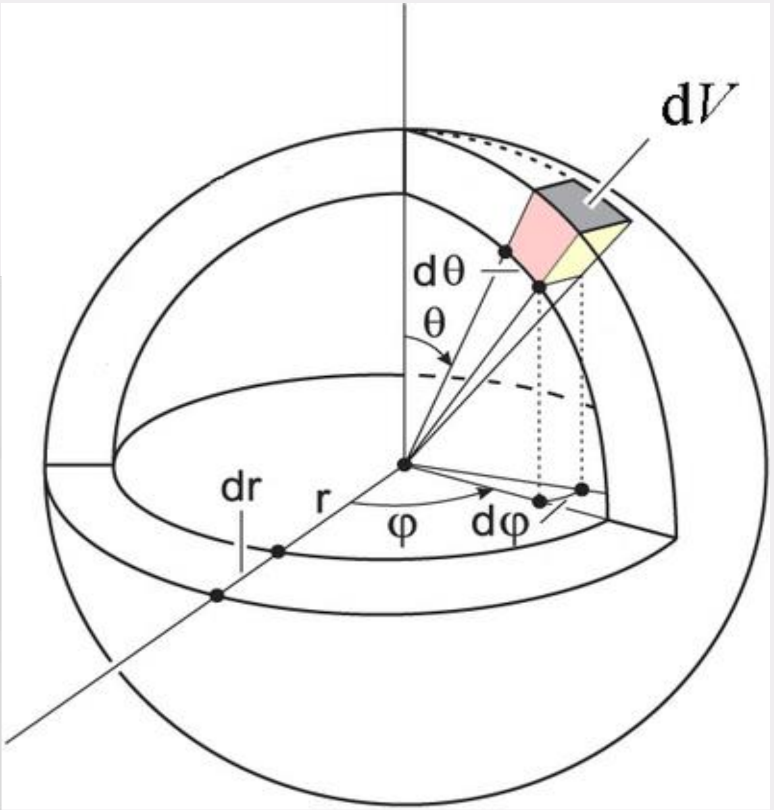
If we fix the radius $\boldsymbol{r}$.

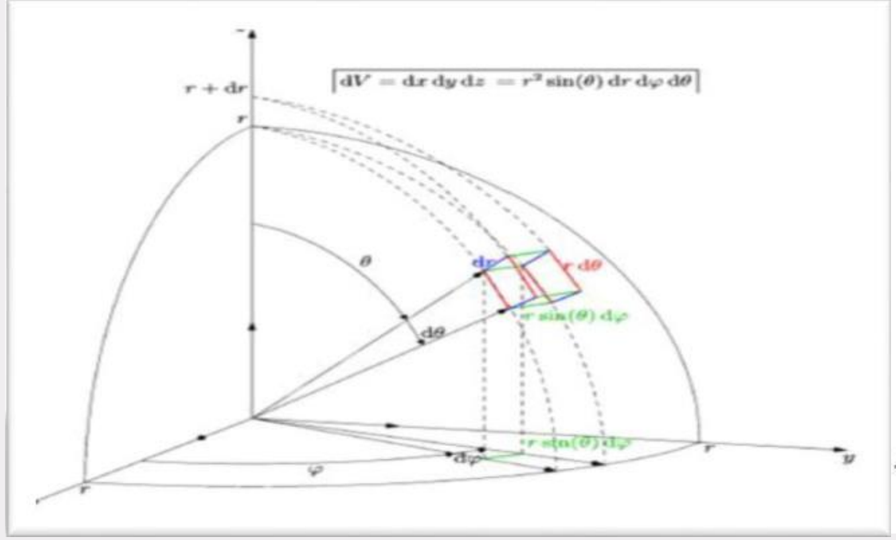
$$dS_\theta = r.\sin\theta.dr.d\varphi$$

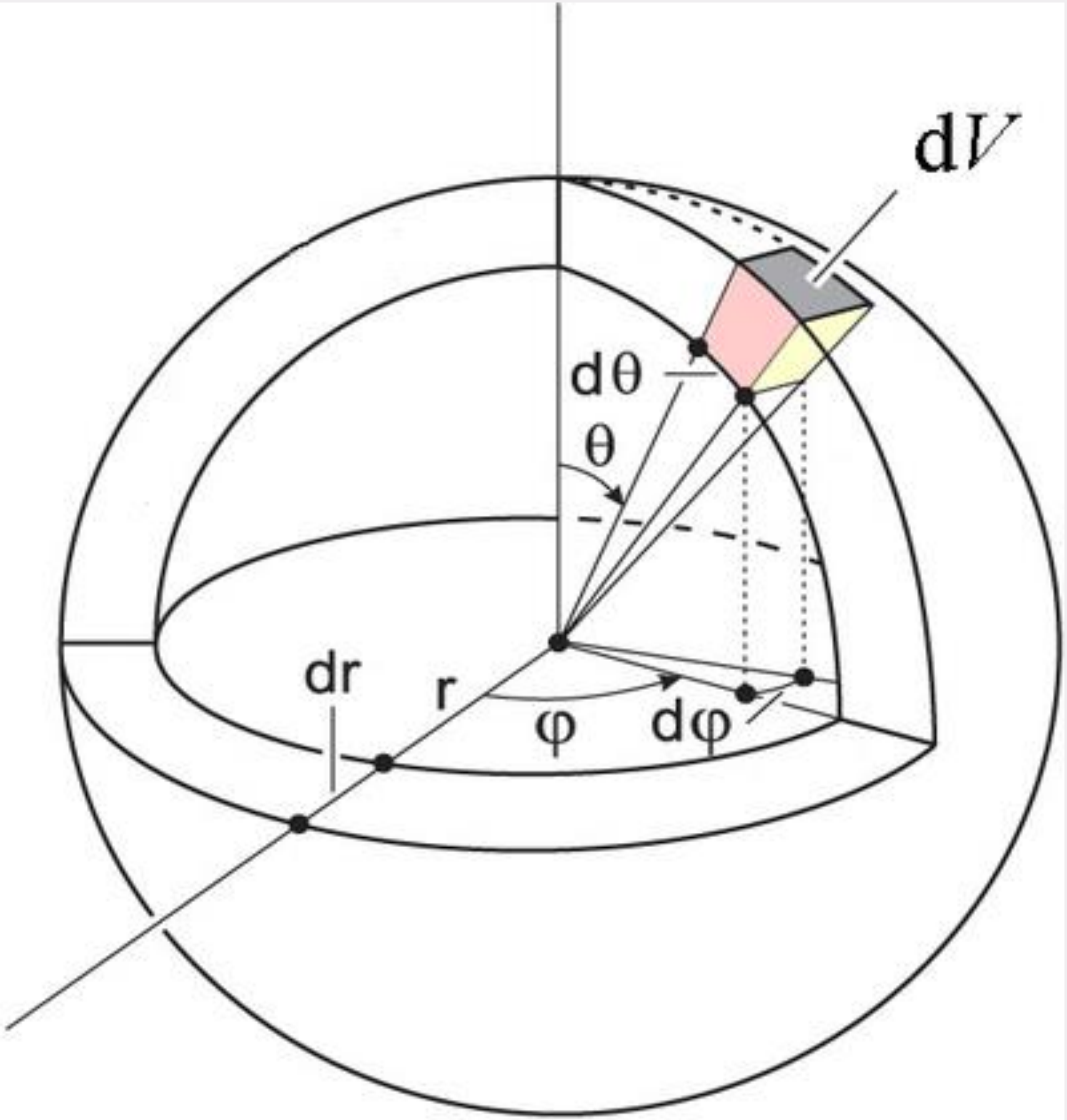
If we fix the angle $\boldsymbol{\theta}$.

$$dS_\varphi = dr.rd\theta$$

If we fix the angle $\boldsymbol{\varphi}$.






dV
dθ
θ
dr
r
φ
dφ

Infinitesimal volume element: We consider the infinitesimal volume ***dV*** generated by the previously described point's ***M*** displacement. This volume is given by:

$$dV = r^2 \sin\theta \, dr d\theta \, d\varphi$$

By varying $\boldsymbol{\varphi}$ from 0 to 2π, $\boldsymbol{r}$ from 0 to R, and $\boldsymbol{\theta}$ from 0 to π, the point will describe a sphere of radius R. We can write:

$$v = \iiint_v dv = \iiint_v r^2 \sin\theta dr d\theta d\varphi$$

$$= \int_0^R r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\varphi = \frac{R^3}{3}.2.2.\pi$$

$$v = \frac{4}{3}\pi R^3$$

We can obtain the surface of the same sphere by fixing $r=R$

$$r = R: \; s = \iint_s ds = R^2 \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\varphi = 4\pi R^2$$

THE OPERATORS: Gradient, divergence, and curl

The Nabla operator $\vec{\nabla}$: It is a vector operator which has the following components: $\left(\frac{\partial}{\partial x} ; \frac{\partial}{\partial y} ; \frac{\partial}{\partial z}\right)$

such that: $\vec{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$

The gradient operator:

Let's define a vector field denoted $\overrightarrow{grad}\, f$ with the expression: $\overrightarrow{grad}\, f = \left(\frac{\partial f}{\partial x} ; \frac{\partial f}{\partial y} ; \frac{\partial f}{\partial z}\right)$

The field $\overrightarrow{grad}\, f$ is called the gradient of *f*. The gradient operator is a vector field acting on a scalar function. $\overrightarrow{grad}\, f = \vec{\nabla}.f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$

The divergence operator:

Let be a vector field $\vec{V}$ (V_x, V_y, V_z). The divergence of $\vec{V}$ is the scalar field denoted $div\,\vec{V}$ and defined as:

$$div\,\vec{V} = \vec{\nabla}.\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

The curl operator:

Let $\vec{V}$ (V_x, V_y, V_z) be a vector field. The curl of $\vec{V}$ is the vector field denoted $\overrightarrow{curl}\,\vec{V}$ and defined as:

$$\overrightarrow{rot\,\vec{V}} = \vec{\nabla} \wedge \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$$= \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\vec{i} - \left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right)\vec{j} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\vec{k}$$