MOHAMED BOUDIAF UNIVERSITY OF MSILA

Faculty of Mathematics and Computer Science **Computer Science Department**

Module : Analysis 1 **University year :**2025/2026

Series N°5. (*) leave from student

Exercise 1 :

1. Solve the following equations in $\mathbb{R}$:

$$1)\quad \arctan(\frac{x}{2}) = \pi, \qquad 2)\quad \arcsin x = \arccos x$$

2. Compute the derivatives of the following functions :

$$f(x) = \arccos \frac{1}{3x}, \quad g(x) = \arcsin \sqrt{x}, \quad h(x) = \arctan(3x^2 + x + 2)(*)$$

3. Show that :

$$a)\ \forall x \in [-1,1],\ \sin(\arccos x) = \sqrt{1-x^2} \qquad b)\ \forall x \in]-1,1[,\ \tan(\arcsin x) = \frac{x}{\sqrt{1-x^2}}$$

$$c)\ \forall x \in [-1,1]\ , \arcsin x + \arccos x = \frac{\pi}{2}, \qquad d)\ \forall x \in \mathbb{R}_+^*\ , \arctan x + \arctan \frac{1}{x} = \frac{\pi}{2}(*)$$

$$e)\ \arctan \alpha + \arctan x = \arctan \frac{x+\alpha}{1-\alpha x}, \quad (\alpha, x) \in \mathbb{R}^2,\ \alpha x \neq 1$$

Exercise 2 :

Let f be a function defined by :

$$f(x) = \begin{cases} \dfrac{\arctan x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases},$$

1. Show that f is differentiable on $\mathbb{R}$, then compute f'.
2. Using the Mean value theorem, prove that : for all $x > 0$; $\quad \arctan x > \dfrac{x}{1+x^2}$.

Exercise 3 : Calculate the derivatives of the following functions :

$$1)\ f(x) = argch(e^x), \quad 2)\ f(x) = th(1+x^2), \quad 3)\ f(x) = argth(\sin x), \quad 4)\ f(x) = \ln th(2x).(*)$$

Prove that :

$$1)\ \forall x \in \mathbb{R},\ th\frac{x}{2} = \frac{sh(x)}{1+ch(x)}, \qquad 2)\ \forall x \neq 0,\ thx = \frac{2}{th(2x)} - \frac{1}{thx}$$

$$3)\quad \forall x \in \mathbb{R},\ argth(\sqrt{\frac{chx-1}{chx+1}}) = \frac{x}{2}$$

Exercise 4 : Prove the following inequalities, valid for all $x \geq 0$.

$$sh(x) \geq x, \qquad ch(x) \geq 1 + \frac{x^2}{2}$$

Exercise 5 :

Prove the following inequalities :

1. $\forall x,\ y \in \mathbb{R}, |\arctan x - \arctan y| \leq |x-y|$
2. $\forall x \geq 0,\ x \leq e^x - 1 \leq xe^x$

Exercise 6 :(*)

— For what values of x do we have $\sqrt{1-x^2} \leq x$?

— Draw up the variation table of the function $x \mapsto \sqrt{1-x^2}.e^{\arcsin x}$.

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0.1 Solution of the series "4"

Exercise 1 :

1. For all $x \in \mathbb{R}$, $\frac{-\pi}{2} \leq \arctan(\frac{x}{2}) \leq \frac{\pi}{2}$, therefore the equation does not solutions.
2. $\arcsin x = \arccos x \Leftrightarrow \sin(\arcsin x) = \sin(\arccos x) \Leftrightarrow x = \sqrt{1-x^2} \Leftrightarrow x = \frac{\sqrt{2}}{2}$
3. The derivatives : direct compute
4. $a)\quad \sin(\arccos x) = \sqrt{1-\cos^2(\arcsin x)} = \sqrt{1-x^2}$
5. $b)\quad \tan(\arcsin x) = \frac{\sin(\arcsin x)}{\cos(\arcsin x)} = \frac{x}{\sqrt{1-x^2}}$
6. $c)\quad \arcsin x + \arccos x = \frac{\pi}{2}$, we can prove this by two methods
   First, consider the function $f(x) = \arcsin x + \arccos x$, continuous on $[-1,1]$, differentiable on $]-1,1[$ and $f'(x) = 0$. Then f is constant over $[-1,1]$, hence
   $f(0) = \arcsin(0) + \arccos(0) = \frac{\pi}{2}$. Therefore $\forall x \in [-1,1],\ f(x) = \frac{\pi}{2}$
   Or, we have : $\arccos x = \frac{\pi}{2} - \arcsin x$, because cos is bijective on $[0,\pi]$ so $\cos(\arccos x) = \cos(\frac{\pi}{2} - \arcsin x)$. So $x = x$, then, $\arcsin x + \arccos x = \frac{\pi}{2}$.
7. d) We have :
   $$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a . \tan b}$$
   we put $a = \arctan\alpha$, $b = \arctan x$, so
   $$\tan(\arctan\alpha + \arctan x) = \frac{\tan\arctan\alpha + \tan\arctan x}{1 - \tan\arctan\alpha . \tan\arctan x} = \frac{\alpha + x}{1 - \alpha . x}$$
   because arctan is a bijective, then : $\quad \arctan\alpha + \arctan x = \arctan(\frac{\alpha + x}{1-\alpha . x})$
8. e) we put $\alpha = \frac{1}{x}$ in precedent formula (d)
   $\arctan\frac{1}{x} + \arctan x = \arctan(\frac{\frac{1}{x} + x}{1 - \frac{1}{x}.x})$
   $$\arctan x + \arctan\frac{1}{x} = \frac{2x+1}{0} = \arctan(\infty) = \frac{\pi}{2}$$

Exercise 2 :

1. If $x \neq 0$, f is quotient of two differentiable functions, then differentiable.
   If $x = 0$, $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{\frac{\arctan x}{x} - 1}{x} = \lim_{x\to 0} \frac{\arctan x - x)}{x^2} = \frac{0}{0}$. I.F By hospital theorem $\lim_{x\to 0} \frac{\frac{1}{1-x^2} - 1}{2x} = \lim_{x\to 0} \frac{-x}{(1+x^2)} = 0$. The f is differentiable at 0, therefore differentiable on $\mathbb{R}$.
   $$f(x) = \begin{cases} \frac{1}{x(1+x^2)} - \frac{\arctan x}{x^2}, & \text{if} \quad x \neq 0 \\ 0, & \text{if} \quad x = 0 \end{cases}$$

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2. we know, $\arctan : \mathbb{R} \to]-\frac{\pi}{2}, \frac{\pi}{2}[$ is continuous on $\mathbb{R}$ then over $[0, x], \quad x > 0$ and differentiable over $]0, x[$, according to mean value theorem there exists $c \in]0, x[$ such that

$$\arctan'(c) = \lim_{x \to 0} \frac{\arctan x - \arctan 0}{x - 0} = \frac{\arctan x}{x} = \frac{1}{1 + c^2}$$

other words, $0 < c < x \Rightarrow 1 < 1 + c^2 < 1 + x^2 \Rightarrow \frac{1}{1 + x^2} < 1 + c^2 < 1$, hence, $\frac{1}{1 + x^2} < \frac{\arctan x}{x}$

$$\text{then,} \quad \frac{x}{1 + x^2} < \arctan x.$$

Exercise 3 :

1. $\forall x \in \mathbb{R} : \dfrac{shx}{1 + chx} = \dfrac{2sh\frac{x}{2}.ch\frac{x}{2}}{2ch^2\frac{x}{2}} = \dfrac{sh\frac{x}{2}}{ch\frac{x}{2}} = th\dfrac{x}{2}.$

2. $\forall x \neq 0 : \dfrac{2}{th2x} - \dfrac{1}{thx} = \dfrac{2}{\frac{2thx}{1 + th^2x}} - \dfrac{1}{thx} = \dfrac{th^2x}{thx} = thx.$

3. We have $thx = \dfrac{2th\frac{x}{2}}{1 + th^2\frac{x}{2}}$ and

$$chx - 1 = ch^2\frac{x}{2} + sh^2\frac{x}{2} - ch^2\frac{x}{2} + sh^2\frac{x}{2} = 2sh^2\frac{x}{2}, \quad chx + 1 = ch^2\frac{x}{2} + sh^2\frac{x}{2} + ch^2\frac{x}{2} - sh^2\frac{x}{2} = 2ch^2\frac{x}{2}.$$

So

$$\sqrt{\frac{chx - 1}{chx + 1}} = \sqrt{th^2\frac{x}{2}} \quad \text{then} \quad argth(th\frac{x}{2}) = \frac{x}{2}.$$

Exercise 4 :

For $sh(x) \geq x$ let : $f(x) = sh(x) - x$ so $f'(x) = ch(x) - 1, \quad x \geq 0$
for every $x \geq 0$, $f'(x) \geq 0$ then f is increasing and $f(0) = 0$, then
$f(x) \geq 0 \Leftrightarrow f(x) - x \geq 0 \Leftrightarrow f(x) \geq x$.

For : $ch(x) \geq 1 + \dfrac{x^2}{2}$

Let $g(x) = ch(x) - 1 - \dfrac{x^2}{2}$ we have $g(0) = 0$ et $g'(x) = sh(x) - x \geq 0$ then g is increasing and $g(0) = 0$, therefore $g(x) \geq 0 \Leftrightarrow ch(x) \geq 1 + \dfrac{x^2}{2}$

Exercise 5 :

1. According to the inequality of Main value theorem i.e
   (f differentiable function on D. if $\forall x, y \in D : |f'(x)| \leq M$:) then

$$|f(x) - f(y)| \leq |x - y|$$

   The function arctan is differentiable on $\mathbb{R}$ and its derivative : $\arctan' x = \dfrac{1}{1 + x^2}$,
   so

$$|\frac{1}{1 + x^2}| \leq 1,$$

   then

$$\forall x, y \in \mathbb{R} : |\arcsin x - \arctan y| \leq |x - y|$$

2. We apply the Main value theorem on $[0, x]$ on we obtain
   $e^x - e^0 = e^c(x - 0) \Leftrightarrow e^x - 1 = x.e^c$, or we have $0 \leq c \leq x \Leftrightarrow x \leq xe^c \leq xe^x$. Then

$$x \leq e^x - 1 \leq xe^x$$

Exercise 6 (*) :

1. For $\sqrt{1-x^2} \leq x$ has meaning if, $x \geq 0$ and $1-x^2 \geq 0$ hence $0 \leq x \leq 1$, so

$$1-x^2 \leq x^2 \Leftrightarrow x^2 \geq \frac{1}{2} \Leftrightarrow x \in [\frac{\sqrt{2}}{2}, 1].$$

2. f defined on $[-1,1]$ and $f'(x) = \dfrac{-x+\sqrt{1-x^2}}{\sqrt{1-x^2}} e^{\arcsin x}$

   $f'(x) = 0 \Leftrightarrow x = \dfrac{\sqrt{2}}{2}$,

$$\begin{cases} f'(x) > 0, \text{ if } x \in]-1, \frac{\sqrt{2}}{2}[, & f \text{ is increasing} \\ f'(x) < 0, \text{ if } x \in]\frac{\sqrt{2}}{2}, 1[& f \text{ is decreasing} \end{cases}$$

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