MOHAMED BOUDIAF UNIVERSITY OF MSILA

Faculty of Mathematics and Computer Science **Computer Science Department**

Module : Analysis 1 **University year :**2025/2026

Series N°4

Remark : (*) leave to the student

Exercise 1 : Find the domain (maximal domain of the real numbers) of the following functions :

1) $\sqrt{2x^2-12x+18}$, **(*)** 2) $\ln(x^2+4x+4)$ 3) $\sqrt{\dfrac{8-16x}{(7+x)^2}}$, **(*)** 4) $\ln(3-x)+\dfrac{\sqrt{x-1}}{x-2}$

Exercise 2 : $f:\mathbb{R}\to\mathbb{R}$, is even function. We assume that the restriction from f to $\mathbb{R}_-$ is increasing. What can we say about the monotony of the restriction of f to $\mathbb{R}_+$.

Exercise 3 : Let $f, g:\mathbb{R}\to\mathbb{R}$, two odd functions. What can we say about $f+g$, $f.g$, and $f\circ g$

Exercise 4 (*) : Let f real function defined by : $f(x)=\ln(|\sin(\frac{\pi}{2}x)|)$.
What is the domain of definition of f ? the function f is even ? odd ? periodic ?

Exercise 5 : We use the definition of limit, show that

$$\lim_{x\to 1}(5x+3)=8\textbf{(*)},\quad \lim_{x\to+\infty}\frac{x+1}{x-1}=1,\quad \lim_{x\to 2}\frac{3}{(x-2)^2}=+\infty$$

Exercise 6 : Calculate the following limits :

$$1)\ \lim_{x\to 1}\left(\frac{1}{1-x}-\frac{2}{1-x^2}\right),\quad 2)\ \lim_{x\to+\infty}\left(1+\frac{y}{e^x}\right)^{e^x}\quad (y\neq 0),\quad 3)\ \lim_{x\to 0^+}\left(\sqrt{x}\sin\frac{1}{\sqrt{x}}\right),$$

$$4)\lim_{x\to 3}\left(\frac{\sqrt{x+1}-2}{x-3}\right),\quad 5)\lim_{x\to 0}\left(\frac{(1-\cos x)(1+2x)}{x^4-x^2}\right),\quad 6)\lim_{x\to\frac{\pi}{4}}\frac{tgx-1}{x-\frac{\pi}{4}},\textbf{(*)}\quad 7)\lim_{x\to 0}cos\left(\frac{\pi.tgx}{3x}\right)\textbf{(*)}$$

$$8)\ \lim_{x\to 0}\frac{tgx-sinx}{x^3}\textbf{(*)}\quad 10)\ \lim_{x\to 1}\frac{\sqrt[3]{x}-1}{x-1},\quad 11)\ \lim_{x\to 1^+}\frac{\sqrt[4]{x^2-1}}{\sqrt{x-1}}\textbf{(*)}$$

Exercise 7 : Let f and g two functions defined by :

$$f(x)=\frac{ax^3+bx^2+x-1}{x-1},\quad g(x)=\sqrt{4x^2+3x+1}-bx$$

1. Study the limit of f if $|x|\to\infty$.
2. Study the limit of f at 1 (discuss according to a and b.)
3. Find according to the values of b ($b\in\mathbb{R}$) the limit $\lim\limits_{x\to+\infty}g(x)$

Exercise 8 : Let the function f defined by $f(x)=\dfrac{\sqrt{1+x}-(1+ax)}{x^2}$

— Find the finite limit b of f when x tend to zero.
— Deduce that there exist a function ϵ such that

$$\sqrt{1+x}=1+\frac{x}{2}-\frac{x^2}{8}+x^2\epsilon(x)\quad\text{with }\lim_{x\to 0}\epsilon(x)=0.$$

Exercise 9 : Consider the function f defined by the following cases :

$$1-st\begin{cases}\dfrac{1}{\ln|x|};\ \text{if } x\notin\{-1,0,1\}\\ 0\quad \text{if } x=-1,0,1\end{cases};\ 2-nd\begin{cases}\dfrac{\sqrt{x+1}-1}{\tan x};\ \text{if } x\neq 0\\ \dfrac{1}{2}\qquad \text{if } x=0\end{cases};\quad 3-rd\begin{cases}\dfrac{\sin(\pi x)}{x-1};\ \text{if } x\neq 1\\ m \text{ with, } m \text{ is real}\end{cases};$$

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1. (1-st)
   — Precisely the continuity of f at zero (using the definition).
   — It's there continuous at 1 and (-1). What can be conclude about the continuity of f on it's domain.
2. (2-nd) Study the continuity from f to $x_0 = 0$.
3. (3-rd) Find the value of the real m for which f is continuous at $x_0 = 1$.

Exercise 10 : Draw the Graph of following functions and study the continuity

$$f_1(x) = [x]; \qquad f_1(x) = x - [x] \quad ([x] \text{ is the integer part of } x)$$

Exercise 11 : Prove that the following equations admit at least one root in each proposed interval.

1. $\cos x = x$ on $I = [0, \pi]$
2. $1 + \sin x = x$ on $I = [\frac{\pi}{2}, \frac{2\pi}{3}]$

Exercise 12 :(*) Consider the function f continues on $[a, b]$ et x_1, x_2 et x_3 numbers in the interval $[a, b]$. Show that the equation : $3f(x) = f(x_1) + f(x_2) + f(x_3)$ admits at least one solution in $[a, b]$.

Exercise 13 : Using the definition of derived number, find the following limits :

$$1.\ \lim_{x\to 0} \frac{e^{3x+2} - e^2}{x}; \quad 2.\ \lim_{x\to 0} \frac{cosx - 1}{x}(*); \quad 3.\ \lim_{x\to 1} \frac{\ln(2-x)}{x-1}; \quad 4.\ \lim_{x\to \frac{\pi}{2}} \frac{e^{cosx} - 1}{x - \dfrac{\pi}{2}}(*)$$

Exercise 14 : Find the derivatives $f'(x)$ of the following functions

$$f(x) = \frac{\cos x}{1 + \sin x}, (*) \quad f(x) = \sqrt{x}\cos x, \quad f(x) = (1 + \frac{1}{\sqrt[3]{x}})^3, \quad (*) f(x) = \ln(\sqrt{\frac{\sin 2x}{1 - \sin 2x}})$$

Exercise 15 : Let f the real function x defined by

$$\begin{cases} f(x) = \dfrac{\sin^3 \pi|x|}{x^2} & \text{if } x \neq 0 \\ 0 \qquad \text{if } x = 0. \end{cases}$$

Is this function continuous and differentiable for $x = 0$?

Exercise 16 : For which values of b and c are the following functions differentiable at -3 ?

$$f(x) = \begin{cases} (1-x)(2-x) & \text{if } x \geq -3 \\ bx + c & \text{if } x < -3. \end{cases}$$

Exercise 17 Let $f : [0, 1] \to \mathbb{R}$ differentiable such that $f(0) = f(1)$ with f' continues at 0 and at 1.We define g on $[0, 1]$ by

$$g(x) = \begin{cases} f(2x) & \text{if } 0 \leq x \leq \frac{1}{2} \\ f(2x - 1) & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Is g continuous ? differentiable ? If not, what hypothesis must be added for this to be the case ?

Exercise 18 : Verify the Main Value Theorem (M.V.T) for the following functions

$$f(x) = x(x-1), \quad \text{on } [0, 1]; \qquad f(x) = \sin x + 2x, \text{ on } [0, \pi]$$

$$f(x) = \alpha x^2 + \beta x + \gamma, \quad \alpha \neq 0, \quad \text{on } [a, b] \text{ interval of } \mathbb{R}$$

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Solution of Series N°4

Exercise 1 :

2) $D = \mathbb{R} - \{-2\}$, 4) $D = [1, 2[\bigcup]2, 3[$.

Exercise 2 :

Let $0 \leq x \leq y$, then $0 \geq -x \geq -y$. Because f is increasing $f(-x) \geq f(-y)$, and f is even Hence

$$f(x) = f(-x) \geq f(-y) = f(y)$$

Then $f(x) \geq f(y)$, therefore, f is decreasing on $\mathbb{R}^+$.

Exercise 3 :

1. Sum $f + g$:

$$(f + g)(-x) = f(-x) + g(-x) = -f(x) - g(x) = -(f + g)(x), \text{ then f+g is odd.}$$

2. Product $f.g$:

$$(f.g)(-x) = f(-x).g(-x) = -f(x).(-g(x)) = (f.g)(x), \text{ then f+g is even.}$$

3. $f \circ g$:

$$(f \circ g)(-x) = f(g(-x)) = -f(g(x)) = -(f \circ g)(x),$$

then $f \circ g$ is odd.

Exercise 4 :

- $\lim_{x \to +\infty}(\frac{x+1}{x-1}) = 1 \Leftrightarrow \forall \epsilon > 0, \exists B > 0 / \forall x \in \mathbb{R} - \{1\} : x > B \Rightarrow |\frac{x+1}{x-1} - 1| < \epsilon.$

$$|\frac{x+1}{x-1} - 1| < \epsilon \Rightarrow x > \frac{2}{\epsilon} + 1$$

Enough to choose $B = \dfrac{2}{\epsilon} + 1$

- $\lim_{x \to +\infty}(\frac{3}{(x-1)^2}) = +\infty \Leftrightarrow \forall A > 0, \exists \delta > 0 / \forall x \in \mathbb{R} - \{2\} : |x - 2| < \delta \Rightarrow \frac{3}{(x-1)^2} > A.$

Choose $\delta = \sqrt{\frac{3}{A}}$.

Exercise 5 :

1) $\lim_{x \to 1}(\frac{1}{1-x} - \frac{2}{1-x^2}) = -\frac{1}{2}$(*),

2) $\lim_{x \to +\infty}(1 + \frac{y}{e^x})^{e^x} = e^y$, we make change variable $\frac{y}{e^x} = t \quad (x \to +\infty,\ t \to 0)$

3) $\lim_{x \to 0^+}(\sqrt{x} \sin \frac{1}{\sqrt{x}}) = 0$, zero times bounded is zero.

4) $\lim_{x \to 1} \frac{\sqrt[3]{x} - 1}{x - 1} = \lim_{x \to 1} \left(\frac{\sqrt[3]{x} - 1}{(\sqrt[3]{x} - 1)(\sqrt[3]{x}^2 + \sqrt[3]{x} + 1)} \right) = \frac{1}{3}.$

we know $(a^3 - b^3 = (a - b)(a^2 + ab + b^2))$ then $(x - 1 = \sqrt[3]{x} - 1)(\sqrt[3]{x}^2 + \sqrt[3]{x} + 1)$

5) $\lim_{x \to 0}(\frac{(1 - \cos x)(1 + 2x)}{x^4 - x^2}) = \lim_{x \to 0} \frac{1 - \cos x}{x^2} . \frac{1 + 2x}{x^2 - 1} = -\frac{1}{2}.$

6) $\lim_{x \to 0} \frac{tgx - sinx}{x^3} = \lim_{x \to 0} \left(\frac{\sin x}{x} . \frac{1 - \cos x}{x^2} . \frac{1}{\cos x} \right) = \frac{1}{2}.$

Exercise 6 :

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1. limit of f when $|x| \to \infty$
   — If $a = b = 0, \quad \lim_{|x|\to\infty} f(x) = 1.$
   — If $a = 0,\ b \neq 0: \quad \lim_{|x|\to\infty} f(x) \lim_{|x|\to\infty}(bx) = \infty$
   — If $a \neq 0,\ b = 0: \quad \lim_{|x|\to\infty} f(x) = \lim_{|x|\to\infty}(ax^2) = \infty.$
2. limit of f when $|x| \to 1$
   — If
$$\lim_{x\to 1^+} f(x) = \frac{a+b}{0} = \begin{cases} +\infty, & a+b>0, \\ -\infty, & a+b<0. \end{cases}$$
   — If
$$\lim_{x\to 1^-} f(x) = \frac{a+b}{0} = \begin{cases} -\infty, & a+b>0, \\ +\infty, & a+b<0. \end{cases}$$
   — If $a + b = 0 \Rightarrow b = -a$ so If
$$\lim_{x\to 1} f(x) = \lim_{x\to 1}\left(\frac{(x^3 - x^2) + x - 1}{x-1}\right) = \lim_{x\to 1}(ax^2+1) = a+1.$$
3. $\lim_{|x|\to+\infty} g(x) = \lim_{|x|\to+\infty}(\sqrt{4x^2+3x+1} - bx) = \lim_{x\to+\infty}\left(\frac{(4-b^2)x^2+3x+1}{x(\sqrt{(4+\frac{3}{x}+\frac{1}{x^2})}+b)}\right)$
$$= \begin{cases} \frac{3}{b}, & b=-2,\ b=2, \\ +\infty, & b \in]-\infty,-2[, \\ -\infty, & b \in]2,+\infty[, \\ +\infty, & b \in]-2,2[. \end{cases}$$

Exercise 7 :

$$\circ\ f(x) = \frac{[\sqrt{1+x} - (1+ax)][\sqrt{1+x} + (1+ax)]}{x^2[\sqrt{1+x} + (1+ax)]} = \frac{(1+x) - (1+ax)^2}{x^2[\sqrt{1+x} + (1+ax)]}$$
$$= \frac{-a^2x^2 + (1-2a)x}{x^2[\sqrt{1+x} + (1+ax)]} = \frac{-a^2}{\sqrt{1+x} + (1+ax)} + \frac{1-2a}{x(\sqrt{1+x} + (1+ax))}$$

In this latter form, it immediately appears that f does not finite limit b, when x tends to zero if we have $1 - 2a = 0$ then $a = \frac{1}{2}$ end in this cases
$$\lim_{x\to 0} f(x) = b = -\frac{a^2}{2} = -\frac{1}{8}.$$

$\circ$ Find the ϵ function

We deduce about the precedent compute that
$$\lim_{x\to 0}\frac{\sqrt{1+x} - (1+\frac{x}{2})}{x^2} = -\frac{1}{8}$$

The function ϵ defined by
$$\epsilon(x) = \frac{\sqrt{1+x} - (1+\frac{x}{2})}{x^2} + \frac{1}{8},$$
such that $\lim_{x\to 0}(\epsilon(x)) = 0$ then
$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + x^2\epsilon(x).$$

Exercise 8 :

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1. (1-st).
— Continuity at points x with $|x| \neq 1$ and $x \neq 0$.

$x \mapsto \dfrac{1}{1 \ln|x|}$ is a composition and inverse of continuous functions ($x \mapsto \ln|x|$).

— Continuity at $x = 0$.

We compute the limit $\lim\limits_{x\to 0} \dfrac{1}{\ln|x|}$. As $x \to 0$ we have $\ln|x| \to -\infty$, hence $\dfrac{1}{\ln|x|} \to 0$. More precisely,

$$\forall \epsilon > 0,\ \exists \delta(\epsilon) > 0 : |x - 0| < \delta(\epsilon) \Rightarrow |(x) - f(0)| < \epsilon.$$

So for given $\varepsilon > 0$ we want $\dfrac{1}{\ln|x|}| < \varepsilon$. This is equivalent to $|\ln|x|| > 1/\varepsilon$, i.e.

$$\ln\frac{1}{|x|} > \frac{1}{\varepsilon} \iff |x| < e^{-1/\varepsilon}.$$

So choosing $\delta = e^{-1/\varepsilon}$ gives $|x| < \delta \implies |\dfrac{1}{\ln|x|}| < \varepsilon$. Therefore

$$\lim_{x\to 0} \frac{1}{\ln|x|} = 0 = f(0),$$

and f is continuous at $x = 0$.

— at $x = 1$ and $x = -1$.

Let $x \to 1$. For $x > 1$ close to 1 we have $\ln|x| > 0$ and $\ln|x| \to 0^+$, so

$$\lim_{x\to 1}\left(\frac{1}{\ln|x|}\right) = +\infty$$

For $0 < x < 1$ close to 1 we have $\ln|x| < 0$ and $\ln|x| \to 0^-$, so

$$\lim_{x\to 1}\left(\frac{1}{\ln|x|}\right) = -\infty$$

f is not continuous at $x = 1$.

The same reasoning applies at $x = -1$.

We conclude that the function f is continuous at every real number except at $x = 1$ and $x = -1$. Equivalently,

$$f \text{ is continuous on } \mathbb{R} \setminus \{-1, 1\}.$$

2. (2-nd) The continuity of f at 0.

$$\lim_{x\to 0} \frac{\sqrt{x+1} - 1}{\tan x} = \lim_{x\to 0} \frac{1}{\frac{\tan x}{x}} . \frac{1}{\sqrt{x+1} + 1} = \frac{1}{2} = f(0).$$

Then f is continuous at 0.

3. The value of the real m

$$\lim_{x\to 1} \frac{\sin(\pi x)}{x - 1} = \frac{0}{0}?$$

By change variable $x - 1 = t$ so

$$\lim_{t\to 1} \frac{\sin(\pi t + \pi)}{t} = -\lim_{t\to 0} \frac{\sin(\pi t)}{\pi t} = -\pi$$

f is continuous at 1, i.e $\lim_{t\to 1} f(x) = f(1) \Rightarrow m = -\pi$

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Exercise 9 :

$$f_1(x) = [x] = \begin{cases} ---\\ ---\\ -2, \quad [-2,-1[,\\ -1, \quad [-1,0[,\\ 0, \quad [0,1[,\\ 1, \quad [1,2[,\\ ---\\ ----\end{cases}$$

is piecewise continuous function, but is not continuous at any integer $(a \in \mathbb{Z}.)$

Ended

$$\lim_{t\to a^-} f_1(x) \neq \lim_{t\to a^+} f_1(x)$$

Is the same from f_2

$$f_2(x) = x - [x] = \begin{cases} ---\\ ---\\ x-2, \quad [-2,-1[,\\ x-1, \quad [-1,0[,\\ x, \quad [0,1[,\\ x+1, \quad [1,2[,\\ ---\\ ----\end{cases}$$

Exercise 10 :

1.

$$\cos x = x \Leftrightarrow \cos x - x = 0$$

Put $f(x) = \cos x - x$ is continuous on $\mathbb{R}$ (because deference two continuous functions $x \mapsto \cos x, \quad x \mapsto x.$)

So f is continuous on $[0, \pi]$, also

$$f(0) = 1, \quad f(\pi) = -1 - \pi \Rightarrow f(0).f(\pi) < 0,$$

according to the intermediate value theorem, there exist $c \in]0, \pi[$ such that : $f(c) = 0$.

2. is the same way

Put $f(x) = 1 + \sin x - x$ is continuous on $\mathbb{R}$ (because is sum and deference two continuous functions $x \mapsto 1$, $x \mapsto \cos x$, and $x \mapsto x$.)

$$f(\frac{\pi}{2}).f(\frac{2\pi}{3}) < 0,$$

according to the intermediate value theorem, there exist $c \in]\frac{\pi}{2}, \frac{2\pi}{3}[$ such that : $f(c) = 0$.

Exercise 11 :

Take $f(x) = e^{3x+2}$ so

$$\lim_{t\to 0}\left(\frac{f(x)-f(0)}{x-0} = f'(0) = 3e^2\right)$$

Also

$$\lim_{t\to 1}\left(\frac{\ln(2-x)-\ln(2-1)}{x-1}\right) = [\ln(2-x)]'(1) = -1$$

Exercise 12 : Clear

Exercise 13 :

At first find the values of b and c such that the function f is continuous at -3.

$$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+}(1-x)(2-x) = 20$$

$$\lim_{x\to -3^-} f(x) = \lim_{x\to -3^-}(bx+c) = -3b+c$$

Therefore f is continuous at -3 if and only if

$$-3b+c = 20$$

Otherwise, the function f is differentiable at -3

The left hand side and right hand side limit are equal

$$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+}\frac{(1-x)(2-x)-((1-x)(2-x))(-3)}{x+3} = ((1-x)(2-x))'(-3) = -9$$

$$\lim_{x\to -3^-} f(x) = \lim_{x\to -3^+}\frac{bx+c-20}{x+3} = (bx+c)'(-3) = b$$

Then $b = -9$, and $-3(-9)+c = 20 \Rightarrow c = -7$.

Exercise 14 : g is differentiable on $]0, \frac{1}{2}[$ because it's composed of two differentiable functions (f and $x \mapsto 2x$,) hence continuous (is the same on $]\frac{1}{2}, 1[$)

- The continuous at $\dfrac{1}{2}$

$$\lim_{x\to \frac{1}{2}^-} g(x) = f(2(\frac{1}{2})) = f(1) = f(0) = \lim_{x\to \frac{1}{2}^+} g(x) = g(\frac{1}{2})$$

So g is continuous at $\dfrac{1}{2}$.

- The differentiable of g at $\dfrac{1}{2}$

We have, for $x < \dfrac{1}{2}$

$g'(x) = 2f'(2x)$ so

$$\lim_{x\to \frac{1}{2}^+}\frac{g(x)-g(\frac{1}{2})}{x-\frac{1}{2}} = g'(\frac{1}{2}) = 2f'(1)$$

hence g has a left derivative at $\dfrac{1}{2}$ equal $2f'(1)$ and the same, g has a right derivative at $\dfrac{1}{2}$ equal $2f'(0)$.

Then g is differentiable at $\dfrac{1}{2}$ if $g'_<(\frac{1}{2}) = g'_>(\frac{1}{2})$ i.e $2f'(0) = 2f'(1)$ if and only if

$$f'(0) = f'(1).$$

Exercise 14 :

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1. f is continuous on $[0,1]$ and differentiable on $]0,1[$, according to the Main Value Theorem

$$\exists c \in]0,1[: f(1) - f(0) = (1-0)f'(c) \Rightarrow c = \frac{1}{2}.$$

2. Under the same conditions of (M.V.T)

$$\exists c \in]0,1[: f(\pi) - f(0) = (\pi-0)f'(c) \Rightarrow c = \frac{\pi}{2}.$$

3. $c = \dfrac{a+b}{2}$

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