Series N°3 : The Sequences

Exercise 1 : Study the monotony of the sequences $(u_n)_{n\geq 1}$ defined as follows.

(a) $u_n = (n+1)(n+2)...(n+n)$
(b) $u_n = na + (-1)^n$ (a positive real)
(c) $u_n = n^n - n!$
(d) $u_n = n^2 - e^n$

Exercise 2 : Give an example in each following situations

1. A positive decreasing sequence whose general term (or n-th term) does not tends to zero.
2. A bounded sequence non-convergent.
3. non-bounded positive sequence not tending towards $+\infty$
4. A non-monotonic sequence that tends to zero.
5. A positive sequence that tends to 0, and that not decreasing.

Exercise 3 : We use the definition of limit, show that :

$$\lim_{n\to+\infty} \frac{2n+1}{n+3} = 2, \quad \lim_{n\to+\infty} \sqrt{n+1} - \sqrt{n} = 0, \quad \lim_{n\to+\infty} 2^{n^2} = +\infty,$$

Exercise 4 : Compute the limits of the following sequences

$$u_n = \sqrt{n^2+n} - n, \qquad u_n = \frac{sin(n^2)}{n}, \qquad u_n = (a)^{\frac{1}{n}} \ (a > 0),$$

$$u_n = \frac{n+(-1)^n}{2n+(-1)^n}, \quad u_n = \frac{1+2+...+n}{n^2}, \quad u_n = \frac{1^2+2^2+...+n^2}{n^3}, \quad u_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}$$

$$u_n = \frac{sin(n)+3cos(n^2)}{\sqrt{n}}, \quad u_n = \frac{2n+(-1)^n}{5n+(-1)^{n+1}}, \quad \frac{n^3+5n}{4n^2+sin(n)+\ln(n)}, \quad u_n = 3^n e^{-3n}$$

(***Indication***. $1+2^2+...+n^2 = \frac{1}{6}n(n+1)(2n+1)$)

Exercise 5 : Let (u_n) and (v_n) be the two sequences defined for $n \geq 1$ as follow

$$u_n = \sum_{k=0}^{n} \frac{1}{k!}, \quad v_n = u_n + \frac{1}{n.n!}$$

1. Prove that the sequences (u_n) and (v_n) are adjacent.
2. We note by e their common limit. Show that for all $n \in \mathbb{N}^*$,

$$n!u_n < n!e < n!u_n + \frac{1}{n}.$$

3. Deduce that e is irrational number.

Exercise 6 : We consider the following sequence

$$u_1 = a, \quad u_2 = b, \quad u_3 = \frac{1}{2}(u_1+u_2)..., \quad u_n = \frac{1}{2}(u_{n-1}+u_{n-2}), ...$$

— Show that the sequence (v_n) defined by the n-th term $v_n = u_n - u_{n-1}$ is geometric sequence.
— Calculate $v_2 + v_3 + ... + v_n$ in function of u_n and a.
— Deduce u_n in function of a, b and n then the limit of u_n when n tends to $+\infty$.

Exercise 7 : We consider the sequence (u_n) define by

$$\begin{cases} u_1 = \sqrt{2} \\ u_{n+1} = \sqrt{2 + u_n}, \quad n \in \mathbb{N}^* \end{cases}$$

1. Calculate u_n for $n = 1, 2, ..., 10$. (Leave it to the student)
2. Prove, by induction demonstrate, that this sequence is bounded above by the number 2.
3. Show that $2 - u_{n+1} < \dfrac{2 - u_n}{2}$. Deduce that the sequence u_n is convergent to 2.

Exercise 8 : We consider u_n defined by

$$(1) \qquad u_n = 3u_{n-1} - n2^n, \qquad \text{and} \qquad u_0 = 9.$$

1°. Determine a and b real such that the sequence (v_n), of general term $v_n = (an + b)2^n$, is solution of the equation (1).
2°. The real numbers a and b having the values found in the previous question, calculate $u_n - v_n$ in function of $u_{n-1} - v_{n-1}$. Deduce the expression of $u_n - v_n$ then that of u_n in function of n.

Exercise 9.Homework : (u_n) sequence defined as follow

$$u_{n+1} = f(u_n) = \frac{7u_n - 10}{u_n}, \qquad \text{and } u_0 \text{ is 1st term}$$

1° Show that there exists two values α and β of u_0 such that the sequence is stationary from u_0.
2° In all that follows we assume that $u_0 = 3$. Calculate $u_{n+1} - \alpha$ et $u_{n+1} - \beta$ in function u_n, deduce that (u_n) is bounded by numbers, which will be specified.
3° Compute $u_{n+1} - u_n$ in function of u_n and deduce that the sequence (u_n) is increasing. Show that it is convergent and calculate its limit l.
4° Proof that we have

$$\frac{u_{n+1} - 2}{u_{n+1} - 5} = \frac{5}{2} \times \frac{u_n - 2}{u_n - 5},$$

deduce u_n in function of n and find the previous results.

Exercise 10 : Using the Cauchy criterion, prove that the sequence of general term $u_n = (-1)^n$ is not convergent.

Exercise 11 :

1. Prove that the sequence of general term

$$u_n = \sum_{k=1}^{n} \frac{1}{k},$$

is not from Cauchy. (Ind. Chose $p = N$and $q = 2N$, for all $N \in \mathbb{N}^*$.)
2. What can we deduce ?
3. Show that (u_n) is increasing and deduce its limit.
4. Same questions from

$$v_n = \sum_{k=2}^{n} \frac{1}{\ln k} \qquad \text{(Leave it to the student)}$$

5. The sequence

$$u_n = \sum_{k=1}^{n} \frac{1}{k^2},$$

is from Cauchy ?

Solutions — Series N°3 : The Sequences

Exercise 1 :

—

$$\frac{u_{n+1}}{u_n} = \frac{(n+2)(n+3)...(2n+2)}{(n+1)....(n+n)} = 2 > 1$$

—

$$u_{n+1} - u_n = a + 2(-1)^{n+1} \begin{cases} a+2 & \text{if } n \text{ odd} \\ a-2 & \text{if } n \text{ even} \end{cases}$$

If $n \geq 2,\ u_{n+1} - u_n$ is always positive and (u_n) is increasing

If $0 \leq a < 2$ so $2(-1)^{n+1}$ equal

$$\begin{cases} a+2 > 0 & \text{if } n \text{ odd} \\ a-2 < 0 & \text{if } n \text{ even.} \end{cases}$$

Then, the sequence (u_n) is neither increasing nor decreasing.

—

$$u_{n+1} - u_n = (n+1)^{n+1} - (n+1)! - n^n + n! = (n+1)^{n+1} - n!n - n^n$$

Because $n! \leq n^n$ then

$$u_{n+1} - u_n = (n+1)^{n+1} - nn^n - n^n \geq (n+1)^{n+1} - nn^n(n+1) = (n+1)((n+1)^{n+1} - n^n).$$

Then (u_n) is increasing.

— We consider the function f on $\mathbb{R}^+$ defined by $f(x) = x^2 - e^x$ so $f'(x) = 2x - e^x$

f' attained its maximum at $x = \ln 2$, so $f'(\ln 2) = 2\ln 2 - 2 < 0$ and f' is negative and f is decreasing, hence the sequence is also decreasing.

Exercise 2 : For example,

1. The sequence that the n^{th} is $u_n = 1 + \frac{1}{n}, \quad n \geq 1$ (positive terms and decreasing sequence).
2. The sequence defined by the n^{th} term $u_n = (-1)^n$
3. u_n sequence defined by

$$\begin{cases} u_{2n} = n \\ u_{2n+1} = 1 \end{cases}$$

Positive terms and not tends to $+\infty$

4. The sequence defined by the n-th term $u_n = \frac{(-1^n)}{n}$
5. Let $u_{2n} = \frac{1}{n}$ and $u_{2n+1} = \frac{2}{n}$ for $n \geq 1$ positive, tends to zero and not decreasing

Exercise 3 :

We must prove the following limits using the definition.

1. $\lim\limits_{n\to\infty} \frac{2n+1}{n+3} = 2 \Leftrightarrow \forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n > N \Rightarrow |u_n - 2| < \epsilon$. We chose $N = \left[\frac{5}{\epsilon} - 3\right] + 1$

   Hence $\forall n \geq N \Rightarrow |u_n - 2| < \epsilon$.
2. $\lim\limits_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right) = 0 \Leftrightarrow \forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n > N \Rightarrow |\sqrt{n+1} - \sqrt{n}| < \epsilon$. We chose $N = \left[\frac{1}{4\epsilon^2} - 3\right] + 1$
3. $\lim\limits_{n\to\infty} 2n^2 = +\infty \Leftrightarrow \forall A > 0, \exists N \in \mathbb{N}, \forall n > N \Rightarrow u_n > A$. We chose $N = \left[\sqrt{\frac{\ln A}{\ln 2}}\right] + 1$

Exercise 4 :

1. $u_n = \sqrt{n^2+n} - n$.
$$\lim_{n\to\infty}(\sqrt{n^2+n}-n) = \lim_{n\to\infty}\Big(\frac{1}{\sqrt{1+\frac{1}{n}}+1}\Big) = \frac{1}{2}.$$

2. $u_n = \dfrac{\sin(n^2)}{n}$.
$|\sin(n^2)| \le 1$ so $|u_n| \le 1/n \to 0$. Hence $u_n \to 0$.

3. $u_n = a^{1/n}$ with $a > 0$.
$a^{1/n} = e^{(\ln a)/n} \to e^0 = 1$.

4. $\lim_{n\to\infty}\Big(\dfrac{n+(-1)^n}{2n+(-1)^n}\Big) = \dfrac{1}{2}$.

5. $u_n = \dfrac{1+2+\cdots+n}{n^2}$.
$1+2+\cdots+n = \dfrac{n(n+1)}{2}$ so
$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\Big(\frac{n(n+1)}{2n^2}\Big) = \frac{1}{2}$$

6. $u_n = \dfrac{1^2+2^2+\cdots+n^2}{n^3}$.
use formula $1^2+\cdots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$. Then
$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\Big(\frac{n(n+1)(2n+1)}{6n^3}\Big) = \frac{1}{3}.$$

7. $u_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}$.
We have : $\dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$. Hence
$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\Big(1-\frac{1}{n+1}\Big) = 1.$$

8. $u_n = \dfrac{\sin(n)+3\cos(n^2)}{\sqrt{n}}$.
Use squeeze theorem $\lim_{n\to\infty} u_n = 0$.

9. $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\Big(\dfrac{2n+(-1)^n}{5n+(-1)^{n+1}}\Big) = \dfrac{2}{5}$.

10. $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\Big(\dfrac{n^3+5n}{4n^2+\sin n+\ln n}\Big) = +\infty$.

11. $\lim_{n\to\infty} u_n = \lim_{n\to\infty}(3ne^{-3n}) = 0$.

Exercise 5 : Let
$$u_n = \sum_{k=0}^{n}\frac{1}{k!}, \qquad v_n = u_n + \frac{1}{n\,n!} \qquad (n \ge 1).$$

1.) Show that (u_n) and (v_n) are adjacent.

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(i) Monotonicity of (u_n) : for every $n \geq 0$,

$$u_{n+1} - u_n = \frac{1}{(n+1)!} > 0,$$

so (u_n) is strictly increasing.

(ii) Monotonicity of (v_n) : compute

$$\begin{aligned} v_{n+1} - v_n &= (u_{n+1} - u_n) + \left(\frac{1}{(n+1)(n+1)!} - \frac{1}{n\,n!} \right) \\ &= \frac{1}{(n+1)!} + \frac{1}{(n+1)(n+1)!} - \frac{1}{n\,n!}. \end{aligned}$$

Factor $1/n!$ to simplify :

$$v_{n+1} - v_n = \frac{1}{n!}\left(\frac{n+2}{(n+1)^2} - \frac{1}{n} \right) = \frac{1}{n!} \cdot \left(\frac{n(n+2) - (n+1)^2}{n(n+1)^2} \right) = -\frac{1}{n!\,n(n+1)^2} < 0.$$

Thus (v_n) is strictly decreasing.

(iii) Order and difference : for all $n \geq 1$, $v_n - u_n = \dfrac{1}{n\,n!} > 0$ and

$$\lim_{n\to\infty} (v_n - u_n) = \lim_{n\to\infty} \frac{1}{n\,n!} = 0.$$

So (u_n) is increasing, (v_n) decreasing, and their difference tends to 0. Therefore (u_n) and (v_n) are adjacent.

2.) e is common limit, we have

$$u_n < e < v_n$$
$$\text{Multiplying by n ! gives}$$
$$u_n n! < en! < v_n n! \Rightarrow u_n n! < en! < u_n n! + \frac{n!}{nn!}$$

then

$$u_n n! < en! < u_n n! + \frac{1}{n}$$

3.) Deduction : e is irrational

By absurd, let $e = \dfrac{p}{q}$ is rational (p and q) are primes themselves $p \in \mathbb{Z},\ q \in \mathbb{N}^*$
so

$$u_q q! < eq! < u_q q! + \frac{1}{q}$$

An otherwise $q!u_q$ is an integer in $\mathbb{N}^*$.

For all $k \leq q$, we have $\dfrac{q!}{k!} \in \mathbb{N}^*$ and $q!\dfrac{p}{q}$ is an integer in $\mathbb{Z}$

Put $m = u_q q!$ and $s = q!\dfrac{p}{q}$

then $m < s < m + \dfrac{1}{q} \leq m+1$. Contradiction, because s is an integer bounded between two consecutive integers.

Therefore e is irrational.

Exercise 6 :

- Let $u_1 = a, u_2 = b$ and for $n \geq 3$,

$$u_n = \tfrac{1}{2}(u_{n-1} + u_{n-2}).$$

Set $v_n = u_n - u_{n-1}$. Then

$$v_{n+1} = \tfrac{1}{2}(u_n + u_{n-1}) - u_n = -\tfrac{1}{2}(u_n - u_{n-1}) = -\tfrac{1}{2}v_n,$$

so (v_n) is geometric with ratio $-\frac{1}{2}$.

- Compute

$$v_2 + v_3 + ... + v_n = u_n - u_1$$

- Deduce (u_n) in function of a, b and n

$$u_n = (v_2 + v_3 + ... + v_n) + u_1 = \frac{2}{3}(a-b)\Big[(-\frac{1}{2})^{n-1} - 1\Big] + a$$

$$\lim_{n\to\infty} u_n = \frac{2}{3}(b-a) + a.$$

Exercise 7 :

1.

$$u_1 = \sqrt{2} = 1,1442.., \quad u_2 = \sqrt{2+\sqrt{2}} = 1,8477.., \quad u_3 = \sqrt{2+\sqrt{2+\sqrt{2}}} = 1,9616..$$

$$u_4 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}} = 1,9904.., \quad u_5 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}} = 1,997.., \quad ..u_{10} = 1,9999976.$$

2. Prove that $\forall n \in \mathbb{N}^* : u_n \leq 2$. By induction demonstrate $u_1 = \sqrt{2} < 2$ true. We assume that $u_n \leq 2$ and show that $u_{n+1} \leq 2$

$$u_n \leq 2 \Leftrightarrow u_n + 2 \leq 4 \Leftrightarrow \sqrt{u_n + 2} \leq 2 \Leftrightarrow u_{n+1} \leq 2$$

   Then (u_n) is bounded from above by 2.

3.

$$2 - u_{n+1} = 2 - \sqrt{2 + u_n} = \frac{4 - 2 - u_n}{2 + \sqrt{2 + u_n}} = \frac{2 - u_n}{2 + \sqrt{2 + u_n}} < \frac{2 - u_n}{2}.$$

   Therefore

$$\boxed{2 - u_{n+1} < \frac{2 - u_n}{2}}$$

We have $0 < 2 - u_{n+1} < \dfrac{2 - u_n}{2}$

So

$$0 < 2 - u_n < \frac{2 - u_{n-1}}{2} < \frac{2 - u_{n-2}}{2^2} < ... < \frac{2 - u_1}{2^{n-1}}$$

Hence

$$0 < 2 - u_n < \frac{2 - \sqrt{2}}{2^{n-1}} \Leftrightarrow 0 < \lim_{n\to\infty}(2 - u_n) < \lim_{n\to\infty}(\frac{2 - \sqrt{2}}{2^{n-1}})$$

and conclude $u_n \to 2$.

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Exercise 8 :

1. The sequence (v_n) solution of (1) if and only if

$$\forall n : v_n = 3v_{n-1} - n2^n \Leftrightarrow \begin{aligned} &(an+b)2^n = 3(a(n-1)+b)2^{n-1} - n2^n \\ &\Leftrightarrow (a-2)n - 3a + b = 0 \end{aligned}$$

hence $a = 2, \quad b = 6$ and $v_n = 2(n+3)2^n$

2. We have $u_n - v_n = 3(u_{n-1} - v_{n-1})$

   The expression of $u_n - v_n$ in function of n.

   So

$$\begin{aligned} u_n - v_n &= 3(u_{n-1} - v_{n-1}) \\ u_{n-1} - v_{n-1} &= 3(u_{n-2} - v_{n-2}) \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ u_2 - v_2 &= 3(u_1 - v_1) \\ u_1 - v_1 &= 3(u_0 - v_0) \end{aligned}$$

$$\boxed{u_n - v_n = 3^n(u_0 - v_0)}$$

Then

$$u_n = 3^n(9-6) + v_n = \boxed{3^{n+1} + 2^{n+1}(n+3)}$$

Exercise 10 :

Use Cauchy criterion :

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall p > q > N \Rightarrow |u_p - u_q| < \epsilon.$$

take $\epsilon = 1,\ p = 2N+1,\ q = 2N$, then

$$|u_p - u_q| = |(-1)^{2N+1} - (-1)^{2N}| = |-1-1| = 2 > 1$$

Then $u_n = (-1)^n$ is not Cauchy sequence, hence not convergent.

Exercise 11 :

— (1)

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall p > q > N \Rightarrow |u_p - u_q| = |\sum_{k=1}^{p} \frac{1}{k} - \sum_{k=1}^{q} \frac{1}{k}| < \epsilon.$$

$$\begin{aligned} |\sum_{k=1}^{p} \frac{1}{k} - \sum_{k=1}^{q} \frac{1}{k}| &= \frac{1}{q+1} + \frac{1}{q+2} + \ldots + \frac{1}{p} \\ &\geq \frac{1}{p} + \frac{1}{p} + \frac{1}{p} + \ldots + \frac{1}{p} \geq \frac{p-q}{p} \end{aligned}$$

(take $p = 2N, q = N$ and compare partial sums).

$$|u_p - u_q| \geq \frac{2N-N}{2N} = \frac{1}{2}$$

Just chose $\epsilon = \dfrac{1}{2}$

Then for $\epsilon = \dfrac{1}{2} < 0, \exists p = 2N > q = N \geq N$ such that $: |u_p - u_q| \geq \dfrac{1}{2}$.

Therefore the sequence is not from Cauchy.

— (2) We deduce that the sequence is not convergent

— (3) (u_n) is strictly increasing so $\lim_{n\to\infty} u_n = +\infty$.