MOHAMED BOUDIAF UNIVERSITY OF MSILA

Faculty of Mathematics and Computer Science **Computer Science Department**

Module : Analysis 1 **University year :**2025/2026

Series N°2 : Filed of Complex numbers

Exercise 1 :

1. Graphically represent the subsets of $\mathbb{C}$ on the $\mathbb{R}^2$ plane.

$$A = \{z \in \mathbb{C}/Re(z) \geq 1\}, \quad B = \{z \in \mathbb{C}/Im(z) \geq 1\}, \quad C = \{z \in \mathbb{C}/1 \leq |z| \leq 2\}$$

2. Put in trigonometric form, then in exponential form the following complex numbers :

$$z_1 = 1 + i, \quad z_1^2, \quad \frac{z_1}{i}, \quad z_2 = \frac{-4}{1 + i\sqrt{3}}, \quad z_2^3, \quad z_3 = (2 + 2i)^6, \quad z_4 = (\frac{1 + i\sqrt{3}}{1 - i})^{20}.$$

Exercise 2 :

Let $z = (-i)^n, \quad n \in \mathbb{N}$. Prove that

$$z = \begin{cases} 1 \quad \text{if} \quad n = 4k \\ -i \quad \text{if} \quad n = 4k + 1 \\ -1 \quad \text{if} \quad n = 4k + 2 \\ i \quad \text{if} \quad n = 4k + 3 \end{cases}, \quad \text{such that } k \in \mathbb{N}$$

Exercise 3 :

- find the integers $n \in \mathbb{N}$ such that $(1 + i\sqrt{3})^n$ is a real positive number.
- let Z a complex number such that $Z \neq 1$. Prove that $|Z| = 1 \Leftrightarrow \dfrac{1 + Z}{1 - Z} \in i\mathbb{R}$

Exercise 4 : Let the following application

$$F : \mathbb{C} \longrightarrow \mathbb{C}, \; z \mapsto F(z) = z^2 + iz - \frac{1}{4}$$

— Solve in $\mathbb{C}$ the equation $F(z) = 0$

— Let $z_1 = i, \; z_2 = -i$, compute $F(z_1)$ and $F(z_2)$.

— Find the expressions $\overline{F(z)}$ and $F(\overline{z})$. What can ou conclude ? ($\overline{z}$ the conjugate of z)

Exercise 5 :

We search to solve the equation $z^3 + (1 + i)z^2 + (i - 1)z - i = 0$.

1. Search for a pure imaginary solution ia of the equation.
2. Find $b, \; c \in \mathbb{R}$ such that $z^3 + (1 + i)z^2 + (i - 1)z - i = (z - ia)(z^2 + bz + c)$
3. Deduce all solutions of the equation

Exercise :6

1. Let $z \in \mathbb{C}^*$. Prove that $|z| = 1$ if and only if $z = \dfrac{1}{\overline{z}}$
2. Let $a, \; b \in \mathbb{C}$ such that $|a| = |b| = 1$. We assume that $L = \dfrac{a + b}{1 + a.b}, \; K = \dfrac{a + b}{1 - a.b}$.
   Prove that L is a real number and K is an imaginary number.

Exercise : (Homework)

Let $\theta \in \mathbb{R}$. we use the complex number $z = \dfrac{1}{2}e^{i\theta}$, show that :

$$\frac{sin\theta}{2} + \frac{sin2\theta}{4} + \frac{sin3\theta}{8} + ... = \frac{2sin\theta}{5 - 4cos\theta}$$

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Exercise 1 :

1. Graphical representation on $\mathbb{R}^2$ Let $z = x + iy$.

— $A = \{z = x + iy \in \mathbb{C} \mid \Re(z) \geq 1 \mid y \in \mathbb{R}\}$: the closed half-plane to the right of the vertical line $x = 1$ (including the line $x = 1$).

— $B = \{z \in \mathbb{C} \mid \Im(z) \geq 1 \mid x \in \mathbb{R}\}$: the closed half-plane above the horizontal line $y = 1$ (including the line $y = 1$).

— $C = \{z \in \mathbb{C} \mid 1 \leq |z| \leq 2\}$: the closed annulus centered at the origin with inner radius 1, $(x^2 + y^2 \geq 1)$ and outer radius 2, $(x^2 + y^2 \leq 4)$. Points with distance from origin between 1 and 2 inclusive.

2. Trigonometric and exponential forms

We give trigonometric form $r(\cos\theta + i\sin\theta)$ and exponential form $re^{i\theta}$.

1. $z_1 = 1 + i.$ $\quad r = \sqrt{1^2 + 1^2} = \sqrt{2}$, $\theta = \arg z_1 = \frac{\pi}{4}$.

$$z_1 = \sqrt{2}\Big(\cos\tfrac{\pi}{4} + i\sin\tfrac{\pi}{4}\Big) = \sqrt{2}\, e^{i\pi/4}.$$

2. $z_2 = -4/(1 + i\sqrt{3})$

$$1 + i\sqrt{3} = 2\Big(\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}\Big) = 2e^{i\pi/3},$$

so

$$z_2 = -\frac{4}{1 + i\sqrt{3}} = -\frac{4}{2e^{i\pi/3}} = -2e^{-i\pi/3}.$$

3. $z_3 = (2 + 2i)^6$. First $2 + 2i = 2(1 + i) = 2\sqrt{2}\, e^{i\pi/4}$. Thus

$$z_3 = (2\sqrt{2})^6 e^{i6\pi/4} = (2\sqrt{2})^6 e^{i3\pi/2}.$$

$$z_3 = 512\, e^{i3\pi/2} = 512\Big(\cos\tfrac{3\pi}{2} + i\sin\tfrac{3\pi}{2}\Big)$$

4. $\left(\dfrac{1 + i\sqrt{3}}{1 - i}\right)^{20} = 2^{10}e^{i(5\pi/3)} = 2^{10}(\cos(5\pi/3) + i\sin(5\pi/3))$

Because

$$1 + i\sqrt{3} = 2e^{i\pi/3}, \qquad 1 - i = \sqrt{2}\, e^{-i\pi/4},$$

so the quotient is $(2/\sqrt{2})e^{i(\pi/3+\pi/4)} = \sqrt{2}e^{i(7\pi/12)}$

Exercise 2 :

Let $z = (-i)^n$ for $n \in \mathbb{N}$. Note that $-i = e^{-i\pi/2}$ and $(-i)^4 = 1$. Hence the powers are periodic of period 4 :

$$(-i)^{4k} = 1, \qquad (-i)^{4k+1} = -i, \qquad (-i)^{4k+2} = (-i)^2 = (-i)\cdot(-i) = -1,$$
$$(-i)^{4k+3} = (-i)^{4k+2}\cdot(-i) = -1\cdot(-i) = i.$$

Thus

$$(-i)^n = \begin{cases} 1 & \text{if } n = 4k, \\ -i & \text{if } n = 4k+1, \\ -1 & \text{if } n = 4k+2, \\ i & \text{if } n = 4k+3, \end{cases} \qquad k \in \mathbb{N}.$$

Exercise 3 :

— (•) Find integers $n \in \mathbb{N}$ such that $(1 + i\sqrt{3})^n$ is a positive real number.

Compute polar form : $1 + i\sqrt{3} = 2e^{i\pi/3}$. Hence

$$(1 + i\sqrt{3})^n = 2^n e^{in\pi/3} = 2^n\Big(\cos(n\pi/3) + i\sin(n\pi/3)\Big).$$

This number is a positive real iff $e^{in\pi/3} = 1$, i.e. $n\pi/3 \equiv 0 \pmod{2\pi}$.

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Or $(\sin(n\pi/3) = 0)$ and $\cos(n\pi/3) \geq 0$

$$\sin(n\pi/3) = 0 \Leftrightarrow n\pi/3 = k\pi \Leftrightarrow n = 3k$$

but for those values from n we have $\cos(n\pi/3) = \cos(k\pi)$ this is positive iff k is even. Thus n is a multiple of 6 :

$$n = 6k, \qquad k \in \mathbb{N}.$$

— (•) Let $Z \in \mathbb{C}$, $Z \neq 1$. Prove :

$$|Z| = 1 \iff \frac{1+Z}{1-Z} \in i\mathbb{R} \quad \text{(pure imaginary)}.$$

Proof. Assume that $|Z| = 1$ so $Z = e^{i\theta}$; then

$$\frac{1+e^{i\theta}}{1-e^{i\theta}} = \frac{e^{i\theta/2} \cdot 2\cos(\theta/2)}{e^{i\theta/2} \cdot 2i\sin(\theta/2)} = \frac{\cos(\theta/2)}{i\sin(\theta/2)} = -i\frac{\cos(\theta/2)}{\sin(\theta/2)} \in i\mathbb{R}.$$

Second direction.
$\dfrac{1+Z}{1-Z}$ is purely imaginary iff it equals the negative of its complex conjugate. Equivalently

$$\overline{\left(\frac{1+Z}{1-Z}\right)} = -\frac{1+Z}{1-Z}.$$

Compute conjugate :

$$\overline{\left(\frac{1+Z}{1-Z}\right)} = \frac{1+\overline{Z}}{1-\overline{Z}}.$$

Thus the condition becomes

$$\frac{1+\overline{Z}}{1-\overline{Z}} = -\frac{1+Z}{1-Z}.$$

So $(1+\overline{Z})(1-Z) = -(1+Z)(1-\overline{Z}) \Leftrightarrow |Z|^2 = 1$. Hence the equivalence holds.

Or we can prove this implication by other way

$$\frac{1+Z}{1-Z} \in i\mathbb{R} \Leftrightarrow \frac{1+Z}{1-Z} = ia \mid a \in \mathbb{R}$$

We express Z in terms of a hence, we find

$$Z = \frac{-1+ia}{1+ia}$$

then $|Z| = 1$.

Exercise 4 :

—

$$F(z) = 0 \Leftrightarrow z^2 + iz - \frac{1}{4} = 0 \Leftrightarrow (z + i/2)^2 = 0 \Leftrightarrow z = -i/2.$$

— $F(i) = -9/5, \quad F(-i) = -1/4.$

— $\overline{F(z)} = \overline{z}^2 - i\overline{z} - 1/4$ and $F(\overline{z}) = \overline{z}^2 + i\overline{z} - 1/4$ then $\overline{F(z)} \neq F(\overline{z})$

Exercise 5 :

$$z^3 + (1+i)z^2 + (i-1)z - i = 0.$$

1. Find a pure imaginary solution ia Let $z = ia$ $(a \in \mathbb{R})$ and substitute. Shows that $z = -i$ is a root (so $a = -1$).

$$(-i)^3 + (1+i)(-i)^2 + (i-1)(-i) - i = -i(-1) + (1+i)(-1) + (i-1)(-i) - i$$

and simplify to 0.

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2. Factorization and roots Factor the cubic (polynomial division by $z+i$ or use factoring tools) :

$$z^3+(1+i)z^2+(i-1)z-i=(z+i)(z^2+z-1).$$

Thus the remaining factor is $z^2+z-1=0$, whose roots are

$$z=\frac{-1\pm\sqrt{1+4}}{2}=\frac{-1\pm\sqrt{5}}{2}.$$

3. All solutions. Therefore the three solutions are

$$z=-i, \qquad z=\frac{-1+\sqrt{5}}{2}, \qquad z=\frac{-1-\sqrt{5}}{2}.$$

Exercise 6 :

1. For $z\in\mathbb{C}^*$, show $|z|=1 \iff z=\dfrac{1}{\overline{z}}$

   If $|z|=1$ then $z\overline{z}=1$, so $z=1/\overline{z}$.

   Conversely, if $z=1/\overline{z}$ then $z\overline{z}=1$ and hence $|z|=1$.

2. L real $\Leftrightarrow L=\overline{L}$

$$\overline{L}=\overline{(\frac{a+b}{1+ab})}=\frac{\overline{a}.\overline{b}}{1+\overline{a}.\overline{b}}=\frac{\frac{1}{a}+\frac{1}{b}}{1+\frac{1}{a}.\frac{1}{b}}=\frac{a+b}{1+ab}=L$$

3. In the same way we can prove $\overline{K}=-K$

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