MOHAMED BOUDIAF UNIVERSITY OF MSILA
Faculty of Mathematics and Computer Science
Computer Science Department
University year :2025/2026

1.st year bachelor's degree - Semester 1
Final exam in Analysis 1
Date : 18/01/2026 Duration : 2 h

Course questions : (04 Points)

1. Write the negation of Cauchy convergent criterion then use it to prove that the sequence $u_n = (-1)^n$ is not convergent.
2. Let f function defined from $D \subset \mathbb{R}$ into $\mathbb{R}$. Write in mathematical language the following statements :
   a) f is bounded, b) f is odd, c) f is periodic, d) f is increasing.
3. Write the text of Intermediate Value Theorem.

Exercise 1 : (04 Points) Questions 1), 2) and 3) are independents of each other

1. Let a and b be two numbers where $a, b \in \mathbb{Q}^+$ and $\sqrt{ab} \notin \mathbb{Q}^+$. Prove that $\sqrt{a} + \sqrt{b} \notin \mathbb{Q}^+$
2. Find the infimum, supremum, maximum and minimum if they exist of the sets A and B.
$$A = \mathbb{N}^*, \quad B = \{-\frac{5}{x}, \quad 1 \leq x < 5 \text{ and } x \in \mathbb{R}\}.$$
3. Find the modulus and argument of the complex number : $1 + \cos\alpha + i\sin\alpha, \quad \alpha \in]0, \pi[$

Exercise 2 : (06 points) (U_n) and (V_n) be two sequences such that :

$$\begin{cases} U_0 \leq V_0, \quad 0 < \beta < \alpha \\ U_n = \dfrac{\alpha U_{n-1} + \beta V_{n-1}}{\alpha + \beta} \\ V_n = \dfrac{\alpha V_{n-1} + \beta U_{n-1}}{\alpha + \beta} \end{cases}$$

1. Let $W_n = U_n - V_n$. Prove that (W_n) is geometric sequence. Identify the common ratio q and the first term W_0.
2. Prove that (U_n) is increasing and that (V_n) is decreasing.
3. Calculate $\lim(U_n - V_n)$. What do you conclude about (U_n) and (V_n) ?
4. Find the common limit l of (U_n) and (V_n) in function of U_0 and V_0, such that $(U_n + V_n)$ is constant.

Exercise 3 : (06 Points)

1. Using the intermediate value theorem (IVT) to prove that the equation
$$xe^{\sin(x)} = \cos(x),$$
admits a solution in the interval $]0, \frac{\pi}{2}[$.
2. Let f be function defined in $\mathbb{R}$ by
$$f(x) = \begin{cases} x^2 \sin\dfrac{1}{x}, & x > 0 \\ ae^{x^2} - \cos x, & x \leq 0. \end{cases}$$
   (a) Determine the values of a such that f is continuous over $\mathbb{R}$.
   (b) Calculate $\lim_{x\to 0} \dfrac{e^{x^2} - \cos x}{x}$ (you can use L'Hospital's rule).
   (c) Examine the differentiability of f over $\mathbb{R}$.
   (d) Express $f'(x)$ in term of x. (f' the derivative function of f)

0.1 Solution of Analysis 01 exam

Course questions : (04 Pt)

1. **Negation of the Cauchy criterion** for a sequence (u_n)

$$\exists\epsilon > 0/\ \forall N \in \mathbb{N}, \exists m, n \geq N : |u_m - u_n| \geq \epsilon. \quad (0.5)$$

Let $\epsilon = 1$. For any $N \in \mathbb{N}$, take for example $n = N+1$ and $m = N+2$. Then :

$$|U_n - U_m| = |(-1)^{N+1} - (-1)^{N+2}| = |(-1)^N + (-1)^N| = 2 \geq 1. \quad (0.5)$$

So, the negation of the Cauchy criterion is satisfied and the sequence $(-1)^n$ is not convergent.

2. **Mathematical language**

(a) f is bounded :

$$\exists M > 0 \text{ such that } \forall x \in D,\ |f(x)| \leq M. \quad (0.5)$$

(b) f is odd :

$$\forall x \in D,\ (-x \in D \Rightarrow f(-x) = -f(x)). \quad (0.5)$$

(c) f is periodic :

$$\exists T > 0 \text{ such that } \forall x \in D,\ f(x+T) = f(x). \quad (0.5)$$

(d) f is increasing :

$$\forall x_1, x_2 \in D,\ x \leq x_2 \Rightarrow f(x_1) \leq f(x_2). \quad (0.5)$$

3. **Intermediate Value Theorem** (01)

Let $f : [a,b] \to \mathbb{R}$ be a continuous function. For every real number y between $f(a)$ and $f(b)$ i.e

$$f(a) \leq y \leq f(b) \quad \text{or} \quad f(b) \leq y \leq f(a),$$

there exists $c \in [a,b]$ such that

$$f(c) = y.$$

Or

$$f \in \mathcal{C}([a,b]), \text{ and } f(a).f(b) < 0 \Rightarrow \exists c \in]a,b[, \text{ such that } f(c) = 0.$$

Exercise 01 : (04 Pt)

1. Assume that $\sqrt{a} + \sqrt{b} \in \mathbb{Q}^+$, so

$$\sqrt{a} + \sqrt{b})^2 \in \mathbb{Q}^+ \Rightarrow a + b + 2\sqrt{ab} \in \mathbb{Q}^+ \Rightarrow \sqrt{ab} \in \mathbb{Q}^+ \Rightarrow \text{contradiction with assumption}$$

then $\sqrt{ab} \notin \mathbb{Q}^+$. (01)

2. Find the infimum, supremum, maximum and minimum (if they exist) of the sets A, B and C.

$$A = \mathbb{N}^* = \{1, 2, 3, \dots\}$$

- The smallest element is $1 \in \mathbb{N}^*$, hence :

$$\inf A = \min A = 1 \quad (0.5)$$

A is not bounded above, therefore :

$$\sup A = +\infty, \quad \max A \text{ does not exist.} \quad (0.5)$$

$$B = \left\{-\frac{5}{x}\ ;\ 1 \leq x < 5\right\}$$

$$1 \leq x < 5 \Rightarrow -5 \leq \frac{-5}{x} < -1 \Rightarrow B = [-5, -1[$$

Hence :

$$\inf B = \min B = -5, \quad \sup B = -1, \quad \max B \text{ does not exist.} \quad (01)$$

3. Modulus and argument of z

$$z = 2\cos^2\frac{\alpha}{2} + 2i\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} = 2\cos\frac{\alpha}{2}(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}) \quad (0.5)$$

Then

$$|z| = 2\cos\frac{\alpha}{2}, \quad \text{and} \quad \arg(z) = \frac{\alpha}{2}. \quad (0.25) + (0.25)$$

Exercise 3 :

1.) Show that (W_n) is a geometric sequence.

Let $W_n = U_n - V_n$. Then

$$\begin{aligned} W_n &= U_n - V_n \\ &= \frac{\alpha U_{n-1} + \beta V_{n-1}}{\alpha+\beta} - \frac{\alpha V_{n-1} + \beta U_{n-1}}{\alpha+\beta} \\ &= \frac{(\alpha-\beta)(U_{n-1} - V_{n-1})}{\alpha+\beta} \\ &= q\,W_{n-1}, \quad (01) \end{aligned}$$

where

$$q = \frac{\alpha-\beta}{\alpha+\beta}, \qquad W_0 = U_0 - V_0. \quad (0.5) + (0.5)$$

Hence $W_n = q^n W_0$. Since $0 < \beta < \alpha$, we have $0 < q < 1$.

2.) Show that (U_n) is increasing and (V_n) is decreasing.

We compute the successive differences :

$$\begin{aligned} U_n - U_{n-1} &= \frac{\alpha U_{n-1} + \beta V_{n-1} - (\alpha+\beta)U_{n-1}}{\alpha+\beta} = \frac{\beta(V_{n-1} - U_{n-1})}{\alpha+\beta} \\ &= -\frac{\beta}{\alpha+\beta}W_{n-1}. \quad (0.5) \end{aligned}$$

Also,

$$V_n - V_{n-1} = \frac{\beta(U_{n-1} - V_{n-1})}{\alpha+\beta} = \frac{\beta}{\alpha+\beta}W_{n-1}. \quad (0.5)$$

Since $U_0 \le V_0$, we have $W_0 \le 0$. As $0 < q < 1$, $W_n = q^n W_0 \le 0$ for all n. Therefore :

$$U_n - U_{n-1} = -\frac{\beta}{\alpha+\beta}W_{n-1} \ge 0, \quad (0.5) \quad V_n - V_{n-1} = \frac{\beta}{\alpha+\beta}W_{n-1} \le 0. \quad (0.5)$$

Thus, (U_n) is increasing and (V_n) is decreasing.

3.) Calculate $\lim(U_n - V_n)$

$$\lim(U_n - V_n) = \lim q^n W_0 = 0,$$

because $0 < q < 1$, hence $V_n - U_n \to 0$ as $n \to \infty$. (0.5)

We deduce that (U_n) and (V_n) are adjacent (one increasing, the other decreasing, and their difference tends to 0). (0.5)

4.)Find the common limit in terms of U_0 and V_0.

Let $T_n = U_n + V_n$. Then

$$\begin{aligned} T_n &= U_n + V_n \\ &= \frac{\alpha U_{n-1} + \beta V_{n-1}}{\alpha+\beta} + \frac{\alpha V_{n-1} + \beta U_{n-1}}{\alpha+\beta} \\ &= \frac{(\alpha+\beta)(U_{n-1} + V_{n-1})}{\alpha+\beta} = T_{n-1}. \end{aligned}$$

So (T_n) is constant and $T_n = T_0 = U_0 + V_0$ for all n. Since (U_n) and (V_n) are adjacent, they converge to the same limit l. Taking the limit in the equality $T_n = U_n + V_n$, we obtain :

$$2l = U_0 + V_0 \quad \Rightarrow \quad \boxed{l = \frac{U_0 + V_0}{2}}. \quad (01)$$

Exercise 3 : (06,5 Pt)

1. Assume that $g(x) = xe^{\sin x} - \cos x \quad (0.5)$

   We remark that g is continuous on $[0, \frac{\pi}{2}] \quad (0.5)$ and we have

$$g(0) = -1 < 0 \quad \text{and} \quad g(\frac{\pi}{2}) = \frac{\pi e}{2} > 0 \quad (0.25) + (0.25)$$

   According to IVT, g has at least one solution c in $]0, \frac{\pi}{2}[\quad (0.5)$ such that $g(c) = 0 \Leftrightarrow ce^{\sin c} = \cos c. \quad (0.25)$

2. Let the function f

   (a) f is continuous over $]-\infty, 0[\cup]0, +\infty[\quad (0.25)$

$$\lim_{x\to 0^-} (ae^{x^2} - \cos x) = a - 1 = \lim_{x\to 0^+} (x^2 \sin \frac{1}{x}) = 0, \text{ (0 times bounded)} \quad (0.5) + (0.5)$$

   hence $\boxed{a = 1}$. $\quad (0.25)$

   (b)

$$\lim_{x\to 0} \frac{e^{x^2} - \cos x}{x} = \frac{0}{0}, (I.F)$$

   By L'Hopital rule we get

$$\lim_{x\to 0} \frac{(e^{x^2} - \cos x)'}{(x)'} = \lim_{x\to 0} \frac{2xe^{x^2} + \sin x}{1} = 0. \quad (01)$$

   (c) The differentiability of f over $\mathbb{R}$

   f is differentiable over $]-\infty, 0[\cup]0, +\infty[\quad (0.25)$

   At $x = 0$

$$f'(0) = \lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{e^{x^2} - \cos x}{x} = 0, \text{from equation (b)} \quad (0.25)$$

   Since, for $x > 0$

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} x \sin \frac{1}{x} = 0, \text{(0 times bounded)} \quad (0.25)$$

   (d) $f'(x)$ in term of x

$$(0.5) \quad f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x}, & x > 0 \\ 0, & x = 0 \\ 2xe^{x^2} + \sin x, & x < 0. \end{cases}$$