MOHAMED BOUDIAF UNIVERSITY OF MSILA
Faculty of Mathematics and Computer Science
Computer Science Department
University year :2024/2025

1.st year bachelor's degree - Semester 1
Final Exam in Analysis 1
Date : 20/01/2025 Duration : 1 h 30 m

Course questions : (05 Pt)

1. **Let** X **be a bounded subset of** $\mathbb{R}$. **Give the characterization property of supremum** $(\sup X)$ **and infimum** $(\inf X)$.
2. **Using the definition of the limit, prove that the sequence** $U_n = \sqrt{2+\dfrac{1}{n}}$ **is convergent to** $\sqrt{2}$.
3. **Write the text of Main Value Theorem.**

Exercise 1 : (05 Pt)

1. **Let** $(a, b) \in \mathbb{Q}^+ \times \mathbb{Q}^+$ **such that** $\sqrt{ab} \notin \mathbb{Q}$. **Prove that** $\sqrt{a} + 3\sqrt{b} \notin \mathbb{Q}$
2. **Let** X **be a subset of** $\mathbb{R}$ **bounded from above, we define the set** $-X$ **as follow** $-X = \{-x, \quad x \in X\}$. **Prove that :** $\inf(-X) = -\sup(X)$.
3. **Prove that** $\forall x \in [-1, 1] : \tan(\arcsin x) = \dfrac{x}{\sqrt{1-x^2}}$.

Exercise 2 : (03,5 Pt)

(U_n) **numerical sequence defined by**

$$\begin{cases} U_0 = 1, \\ U_{n+1} = U_n \cos(\dfrac{\lambda}{2^{n+1}}), \quad \forall n \in \mathbb{N}, \quad \lambda \in]0, \dfrac{\pi}{2}]. \end{cases}$$

1. **Show that** $\forall n \in \mathbb{N} : U_n > 0$.
2. **Show that the sequence** (U_n) **is decreasing and deduce that it is convergent to a finite limit.**

Exercise 3 : (06,5 Pt)

1. **Using the Main value theorem. Find the following limit**

$$\lim_{x \to +\infty} x^3 (e^{\frac{1}{x^2}} - e^{\frac{1}{(x+1)^2}})$$

2. **Let the real function** $f :]0, +\infty[\to \mathbb{R}$ **defined by the following relation** :

$$f(x) = e^{\frac{x}{2}} - e^{-\frac{x}{2}} - x$$

   (a) **Calculate** $\lim\limits_{x \to +\infty} f(x)$.
   (b) **Calculate** f' **and** f''.
   (c) **Prove that** $f''(x) > 0, \forall x > 0$, **and deduce that** f **is increasing.**
   (d) **Draw up the variation table of** f.

0.1 Solution of Analysis 01 exam

Course questions

1. The characterization of supremum and infimum

$$M = \sup(X) \Leftrightarrow \begin{cases} \forall x \in X : x \leq M \\ \forall \epsilon > 0, \exists x' \in X, M - \epsilon < x' \leq M \end{cases} \quad ...(01)$$

$$m = \inf(X) \Leftrightarrow \begin{cases} \forall x \in X : x \geq m \\ \forall \epsilon > 0, \exists x' \in X, m \leq x' < m + \epsilon \end{cases} \quad ...(01)$$

2. We want to show that

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}^* \ (n > N \Rightarrow |u_n - \sqrt{2}| < \epsilon.) \quad (0,5)$$

Now, we have

$$|u_n - \sqrt{2}| = |\sqrt{2 + \frac{1}{n}} - \sqrt{2}| = \frac{\left(\sqrt{2 + \frac{1}{n}} - \sqrt{2}\right)\left(\sqrt{2 + \frac{1}{n}} + \sqrt{2}\right)}{\sqrt{2 + \frac{1}{n}} + \sqrt{2}} = \frac{\frac{1}{n}}{\sqrt{2 + \frac{1}{n}} + \sqrt{2}}$$

$$\text{and therefore : } |u_n - \sqrt{2}| < \frac{\frac{1}{n}}{\sqrt{2} + \sqrt{2}} = \frac{1}{2\sqrt{2}} \frac{1}{n} ...(01)$$

Consequently, to have, $|u_n - \sqrt{2}| < \epsilon, \epsilon$ chosen arbitrary in $\mathbb{R}_+^*$, we just to have, $\frac{1}{2\sqrt{2}} \frac{1}{n} < \epsilon$,

let $n > \frac{\sqrt{2}}{4\epsilon}$.Then we can take for N any integer greater than $\left[\frac{\sqrt{2}}{4\epsilon}\right] + 1$. ..(0, 5)

3. Main Value Theorem : Let be f continuous on $[a, b]$ and differentiable on $]a, b[$...(0, 5)
There exists a $c \in]a, b[$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} ....(0,5)$$

Exercise 1 :

1. Let $(a, b) \in \mathbb{Q}^+ \times \mathbb{Q}^+$ such that $\sqrt{ab} \notin \mathbb{Q}$.
Suppose that $\sqrt{a} + 3\sqrt{b} \in \mathbb{Q}$....(0, 5), hence, $\exists (p, q) \in \mathbb{Z} \times \mathbb{N}^*$ such that

$$\sqrt{a} + 3\sqrt{b} = \frac{p}{q} ...(0,25)$$

$$(\sqrt{a} + 3\sqrt{b})^2 = \frac{p^2}{q^2} ...(0,25)$$

This implies that :

$$\sqrt{ab} = \frac{\frac{p^2}{q^2} - a - 9b}{6} \in \mathbb{Q}...(0,5)$$

Which is absurd, then $\sqrt{a} + 3\sqrt{b} \notin \mathbb{Q}$....(0, 5),

2. Since X is bounded from above, then $\exists M \in \mathbb{R}$ s.t $M = \sup(X)$ so according to the first course questions, we have

$$M = \sup(X) \Leftrightarrow \begin{cases} \forall x \in X : x \leq M \\ \forall \epsilon > 0, \exists x' \in X, M - \epsilon < x' \leq M \end{cases} \quad ...(0,5)$$

thus

$$-M = -\sup(X) \Leftrightarrow \begin{cases} \forall x \in X : -x \geq -M \\ \forall \epsilon > 0, \exists x' \in X, -M \leq -x' < -M + \epsilon \end{cases} \quad = \inf(-X)...(01)$$

Hence $\inf(-X) = -M = -\sup(X)...(0,5)$

3.

$$\forall x \in [-1,1] : \tan(\arcsin x) = \frac{\sin(arcsinx)}{\cos(arcsinx)} = \frac{x}{\sqrt{1-\sin^2(\arcsin x)}} = \frac{x}{\sqrt{1-x^2}}...(01)$$

Exercise 2 :

1. By induction demonstrate we note that $U_0 = 1 > 0...(0,25)$, so we suppose that $U_n > 0$, and prove that $U_{n+1} > 0....(0,25)$.

   Since $0 < \lambda < \dfrac{\pi}{2}$ and $\dfrac{1}{2^{n+1}} < 1...(0,5)$, this one imply that $\cos(\dfrac{\lambda}{2^{n+1}}) > 0...(0,5)$.

   Thus, $U_{n+1} = U_n \cos(\dfrac{\lambda}{2^{n+1}}) > 0...(0,5)$. Then $\forall n \in \mathbb{N}, U_n > 0$.

2. For all $n \geq 0$.

$$\frac{U_{n+1}}{U_n} = \cos(\frac{\lambda}{2^{n+1}}) < 1....(0,5) \quad \text{for} \quad \lambda \in]0, \frac{\pi}{2}].$$

   This implies that $\forall n \in \mathbb{N}. \quad U_{n+1} < U_n$. Then (U_n) is decreasing...(0,5)

   Thus, we note that (U_n) is bounded from below by 0 and decreasing, then (U_n) is convergent. ....(0,5)

Exercise 3 :

1. Let $f(t) = e^{\frac{1}{t^2}}$. The function f is continuous and differentiable between x and $x+1$ with $x > 0$, we search that there exist $c \in]x, x+1[$ such that :

$$f(x+1) - f(x) = f'(c) \Leftrightarrow e^{\frac{1}{(x+1)^2}} - e^{\frac{1}{x^2}} = \frac{-2}{c^3} e^{\frac{1}{x^2}}...(01)$$

   This one imply that

$$x^3(e^{\frac{1}{x^2}} - e^{\frac{1}{(1+x)^2}}) = \frac{2x^3}{c^3} e^{\frac{1}{x^2}}...(0,25)$$

   Because $0 < x \leq c \leq x+1 \Rightarrow x^3 \leq c^3 \leq (x+1)^3 \Rightarrow \frac{1}{(x+1)^3} \leq \frac{1}{c^3} \leq \frac{1}{x^3}...(0,25)$ We know that the exponential function is increasing, then $e^{\frac{1}{(x+1)^3}} \leq e^{\frac{1}{c^3}} \leq e^{\frac{1}{x^3}}...(0,25)$ Otherwise, we have

$$\frac{1}{(x+1)^3} \leq \frac{1}{c^3} \leq \frac{1}{x^3} \Rightarrow \frac{2x^3}{(x+1)^3} \leq \frac{2x^3}{c^3} \leq \frac{2x^3}{x^3}...(0,25)$$

   So

$$\frac{2x^3}{(x+1)^3} e^{\frac{1}{(x+1)^3}} \leq \frac{2x^3}{c^3} e^{\frac{1}{c^3}} \leq \frac{2x^3}{x^3} e^{\frac{1}{x^3}}...(0,25)$$

   since, we have $\lim\limits_{x \to +\infty} \frac{2}{e}^{\frac{1}{x^3}} = 2$ and $\lim\limits_{x \to +\infty} \frac{2x^3}{(x+1)^3} e^{\frac{1}{(x+1)^3}} = 2$, ...(0,5) then by squeeze theorem, we have

$$\lim_{x \to +\infty} x^3(e^{\frac{1}{x^2}} - e^{\frac{1}{(1+x)^2}}) = 2...(0,25)$$

2.

$$f(x) = e^{\frac{x}{2}} - e^{-\frac{x}{2}} - x$$

(a) $\lim\limits_{x\to+\infty}(e^{\frac{x}{2}} - e^{-\frac{x}{2}} - x) = \lim\limits_{x\to+\infty} e^{\frac{x}{2}}(1 - e^{-x} - \frac{x}{e^{\frac{x}{2}}}) = +\infty$........(0,5)

(b) We have,

$$f'(x) = \frac{1}{2}e^{\frac{x}{2}} + \frac{1}{2}e^{\frac{-x}{2}} - 1....(0,5)$$

$$f''(x) = \frac{1}{4}e^{\frac{x}{2}} - \frac{1}{4}e^{\frac{-x}{2}}....(0,5)$$

(c) We have, $\forall x > 0, e^{\frac{x}{2}} > e^{\frac{-x}{2}}$ which imply that $f''(x) > 0, \forall x > 0$. Thus, f' is increasing on $]0, +\infty[$.....$(0,5)$

Since $f'(0) = 0$, ...$(0,25)$ so for all $x > 0$, we have $f'(x) > 0$. This implies that f is increasing ...(0,5)

(d) Note that $f(0) = 0$....$(0,25)$ then we have the following variation table

x	0	$+\infty$
$f'(x)$	$+$	
$f(x)$	0	$+\infty$

......(0,5)