Inverse trigonometric and Hyperbolic functions

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Inverse trigonometric functions

arccosine function

We assume the function cosine $\cos : \mathbb{R} \to [-1,1], \quad x \mapsto \cos x$. To obtain a bijection from this function, we must consider the restriction of cosine on the interval $[0,\pi]$. On this interval the function **cosine** is continuous and strictly decreasing, then the restriction

$$\cos : \; [0,\pi] \to [-1,1], \quad x \mapsto \cos x.$$

is bijection. Its inverse bijection is the function arccosine :

$$\arccos : \; [-1,1] \to [0,\pi], \quad x \mapsto \arccos x.$$

So we have by definition the inverse bijection :

$$\cos(\arccos x) = x \quad \forall x \in [-1,1]$$

$$\arccos(\cos x) = x \quad \forall x \in [0,\pi]$$

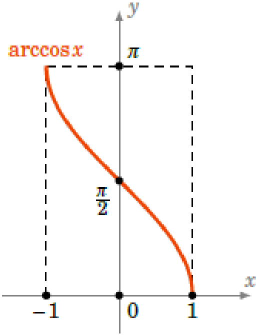
arccosine

In other words :

$$\text{Si} \quad x \in [0, \pi] \qquad \cos x = y \Leftrightarrow x = \arccos y$$

Let's finish with the derivative of arccos :

$$\arccos'(x) = \frac{-1}{\sqrt{1 - x^2}} \qquad \forall x \in]-1, 1[$$



arcsine

The restriction

$$\sin : [\frac{-\pi}{2}, \frac{\pi}{2}] \to [-1, 1], \quad x \mapsto \sin x.$$

is a bijection. Its inverse bijection is the function arcsine :

$$\arcsin : [-1, 1] \to [\frac{-\pi}{2}, \frac{\pi}{2}], \quad x \mapsto \arcsin x.$$

We have by definition :

$$\sin(\arcsin x) = x \quad \forall x \in [-1, 1]$$

$$\arcsin(\sin x) = x \quad \forall x \in [\frac{-\pi}{2}, \frac{\pi}{2}]$$

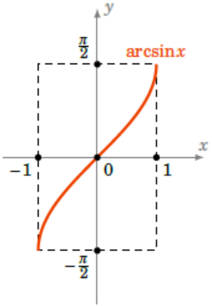
arcsine

In other words :

$$\text{Si} \quad x \in [\frac{-\pi}{2}, \frac{\pi}{2}] \qquad \sin x = y \Leftrightarrow x = \arcsin y$$

Let's finish with the derivative of arcsin is :

$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}} \qquad \forall x \in]-1, 1[$$



arctangent function

The restriction

$$\tan :]\frac{-\pi}{2}, \frac{\pi}{2}[\to \mathbb{R}, \quad x \mapsto \tan x.$$

is a bijection. Its inverse bijection is the function arctangent :

$$\arctan : \mathbb{R} \to]\frac{-\pi}{2}, \frac{\pi}{2}[, \quad x \mapsto \arctan x.$$

$$\tan(\arctan x) = x \quad \forall x \in \mathbb{R}$$

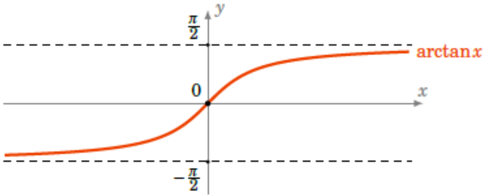
$$\arctan(\tan x) = x \quad \forall x \in]\frac{-\pi}{2}, \frac{\pi}{2}[$$

In other words :

$$\text{Si} \quad x \in]\frac{-\pi}{2}, \frac{\pi}{2}[\qquad \tan x = y \Leftrightarrow x = \arctan y.$$

The derivative of arctan is :

$$\arctan'(x) = \frac{1}{1+x^2} \qquad \forall x \in \mathbb{R}$$



Example

Example

- Find the exact values of

$$\arccos(\frac{1}{2}), \quad \arcsin(-\frac{\sqrt{2}}{2}).$$

- Prove that arcsin is odd function.

Hyperbolic functions and their inverses

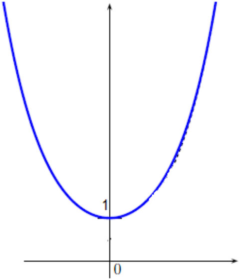
Hyperbolic functions and their inverses

For all $x \in \mathbb{R}$ the **hyperbolic cosine function,** written cosh or ch is defined by the relation :

$$chx = \frac{e^{x} + e^{-x}}{2}.$$

The derivative function is

$$ch'(x) = sh(x)$$



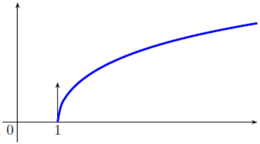
Hyperbolic cosine function and its inverse

The restriction $ch : [0, +\infty[\longrightarrow [1, +\infty[$ is bijective. and its inverse bijection is :

$$argch : [1, +\infty[\longrightarrow [0, +\infty[$$

The derivative function is

$$argch'(x) = \frac{1}{\sqrt{x^2 - 1}}, (\quad x > 1)$$



Hyperbolic sine function and its inverse

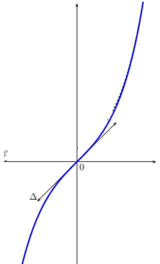
For all $x \in \mathbb{R}$ le **hyperbolic sine function** is defined by the relation :

$$shx = \frac{e^x - e^{-x}}{2}.$$

$sh : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous function, differentiable and strictly increasing verifying

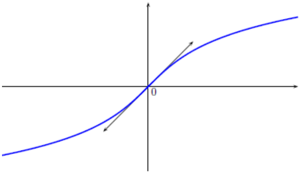
$$\lim_{x \to -\infty} sh(x) = -\infty \text{ et } \lim_{x \to +\infty} sh(x) = +\infty,$$

The derivative function is $sh'(x) = ch(x)$



then is a bijection. And its inverse bijection is : *argsh* : $\mathbb{R} \longrightarrow \mathbb{R}$. The derivative function is :

$$argsh'(x) = \frac{1}{\sqrt{x^2+1}},$$



Remark

The name of these two hyperbolic functions suggest that the have similar properties to the trigonometric functions and some of these will be investigated.

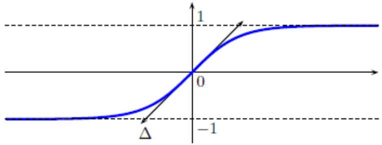
Hyperbolic tangent function and its inverse

By definition the **tangent hyperbolic** is :

$$th(x) = \frac{sh(x)}{ch(x)}.$$

The function $th : \mathbb{R} \longrightarrow]-1, +1[$ is a bijection.

The derivative function is : $th'(x) = 1 - th^2(x) = \dfrac{1}{ch^2(x)}$



Hyperbolic tangent function and its inverse

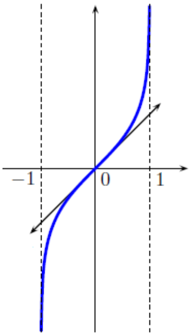
we notice

$$\textcolor{blue}{argth} :]-1,+1[\longrightarrow \mathbb{R}$$

is inverse bijection.

The derivative function is

$$argth'(x) = \frac{1}{1-x^2}, \quad (|x| < 1)$$



Hyperbolic trigonometry

$ch(a+b) = ch(a).ch(b) + sh(a).sh(b), ...(1)$
$ch(a-b) = ch(a).ch(b) - sh(a).sh(b), ...(2)$
$sh(a+b) = sh(a).ch(b) + ch(a).sh(b), ...(3)$
$sh(a-b) = sh(a).ch(b) - ch(a).sh(b)..........(4)$
By multiplying the expression for $(chx + shx)$ and $(chx - shx)$ together, we have

$$ch^2(x) - sh^2(x) = 1$$

obvious,

$$chx + shx = e^x$$

Put $a = b$ in first formulate (1)

$$ch(2a) = ch^2(a) + sh^2(a)$$

Put $a = b$ in third formulate (3)

$$sh(2a) = 2.sh(a).ch(a)$$

Hyperbolic trigonometry

$$th(a+b) = \frac{th(a)+th(b)}{1+th(a).th(b)}.....(5)$$

Put $a = b$ in third formulate (5)

$$th2a = \frac{2tha}{1+th^2a}$$

Remark

Expression of inverse hyperbolic functions with natural logatithm

$$argch(x) = \ln(x+\sqrt{x^2-1}), (\quad x \geq 1)$$

$$argsh(x) = \ln(x+\sqrt{x^2+1}), (\quad x \in \mathbb{R})$$

$$argth(x) = \frac{1}{2}\ln(\frac{1+x}{1-x}), (\quad -1 < x < 1)$$

Exercise

Exercise : Use the definition of chx and shx in term of exponential functions to prove that :

$$ch2x = 2ch^2x - 1, \qquad ch2x = 1 + 2sh^2x$$

Exercise :01

1. $$\forall x \in [-1, 1], \arcsin x + \arccos x = \frac{\pi}{2},$$

2. $$\arctan \alpha + \arctan x = \arctan \frac{x+\alpha}{1-\alpha x}, \quad (\alpha, x) \in \mathbb{R}^2, \alpha x \neq 1$$

3. $$\forall x \in \mathbb{R}_+^*, \arctan x + \arctan \frac{1}{x} = \frac{\pi}{2}$$

4. $$\sin(\arctan x) = \frac{x}{\sqrt{1+x^2}},$$

Solution Exercixe 01

1) We can prove this by two methods
First, consider the function $f(x) = \arcsin x + \arccos x$, continuous on $[-1, 1]$, differentiable on $]-1, 1[$ and $f'(x) = 0$. Then f is constant over $[-1, 1]$, hence
$f(0) = \arcsin(0) + \arccos(0) = \frac{\pi}{2}$. Therefore
$\forall x \in [-1, 1],\ f(x) = \frac{\pi}{2}$
Or, we have :

$$\arcsin x + \arccos x = \frac{\pi}{2} \Leftrightarrow \arccos x = \frac{\pi}{2} - \arcsin x,$$

because cos is bijective on $[0, \pi]$ so
$\cos(\arccos x) = \cos(\frac{\pi}{2} - \arcsin x)$. So $x = x$, then,
$\arcsin x + \arccos x = \frac{\pi}{2}$.

Solution Exercise 01

2) We have :

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a . \tan b}$$

we put $a = \arctan \alpha,\ b = \arctan x$, so

$$\tan(\arctan \alpha + \arctan x) = \frac{\tan \arctan \alpha + \tan \arctan x}{1 - \tan \arctan \alpha . \tan \arctan x} = \frac{\alpha + x}{1 - \alpha . x}$$

because arctan is a bijective, then :

$$\arctan \alpha + \arctan x = \arctan(\frac{\alpha + x}{1 - \alpha . x})$$

Solution Exercise 01

3) We put $\alpha = \dfrac{1}{x}$ in precedent formula

$$\arctan\frac{1}{x} + \arctan x = \arctan\left(\frac{\frac{1}{x} + x}{1 - \frac{1}{x}.x}\right)$$

$$\arctan x + \arctan\frac{1}{x} = \frac{2x+1}{0} = \arctan(\infty) = \frac{\pi}{2}$$

4) We put $y = \arctan x$, we have $\sin^2 y = \dfrac{\tan^2 y}{1+\tan^2 y}$.

So :

$$\sin(\arctan x) = \sqrt{\frac{\tan^2(\arctan x)}{1+\tan^2(\arctan x)}} = \frac{x}{\sqrt{1+x^2}}$$

Exercise 2 :

Prove that :

1. $$\forall x \in \mathbb{R},\ th\frac{x}{2} = \frac{sh(x)}{1 + ch(x)},$$

2. $$\forall x \neq 0,\ thx = \frac{2}{th(2x)} - \frac{1}{thx}$$

3. $$\forall x \in \mathbb{R},\ argth(\sqrt{\frac{chx - 1}{chx + 1}}) = \frac{x}{2}.$$

Solution Exercise 02

1. $\forall x \in \mathbb{R}: \dfrac{shx}{1+chx} = \dfrac{2sh\frac{x}{2}.ch\frac{x}{2}}{2ch^2\frac{x}{2}} = \dfrac{sh\frac{x}{2}}{ch\frac{x}{2}} = th\dfrac{x}{2}.$
2. $\forall x \neq 0: \dfrac{2}{th2x} - \dfrac{1}{thx} = \dfrac{2}{\dfrac{2thx}{1+th^2x}} - \dfrac{1}{thx} = \dfrac{th^2x}{thx} = thx.$
3. We have $thx = \dfrac{2th\frac{x}{2}}{1+th^2\frac{x}{2}}$ and

$$chx - 1 = ch^2\frac{x}{2} + sh^2\frac{x}{2} - ch^2\frac{x}{2} + sh^2\frac{x}{2} = 2sh^2\frac{x}{2},$$
$$chx + 1 = ch^2\frac{x}{2} + sh^2\frac{x}{2} + ch^2\frac{x}{2} - sh^2\frac{x}{2} = 2ch^2\frac{x}{2}.$$

So

$$\sqrt{\frac{chx-1}{chx+1}} = \sqrt{th^2\frac{x}{2}} \quad \text{then} \quad argth(th\frac{x}{2}) = \frac{x}{2}.$$

Exercise 03

1) Prove that, for all $x, y \in \mathbb{R}$,

- $sh(x+y) = sh(x)ch(y) + ch(x)sh(y)$
- $ch(x+y) = ch(x)ch(y) + sh(x)sh(y)$.

2) Let f be a function defined by :

$$f(x) = \begin{cases} \dfrac{\arctan x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases} ,$$

1. Show that f is differentiable on $\mathbb{R}$, then compute f'.
2. Using the Mean value theorem, prove that :
   for all $x > 0$; $\quad \arctan x > \dfrac{x}{1+x^2}$.

Solution Exercise 03

1) If $x \neq 0$, f is quotient of two differentiable functions, then differentiable.

If $x = 0$, $\lim\limits_{x\to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x\to 0} \dfrac{\dfrac{\arctan x}{x} - 1}{x} =$
$\lim\limits_{x\to 0} \dfrac{\arctan x - x)}{x^2} = \dfrac{0}{0}$. I.F By hospital theorem
$\lim\limits_{x\to 0} \dfrac{\dfrac{1}{1 - x^2} - 1}{2x} = \lim\limits_{x\to 0} \dfrac{-x}{(1 + x^2)} = 0$. The f is differentiable at 0, therefore differentiable on $\mathbb{R}$.

$$f'(x) = \begin{cases} \dfrac{1}{x(1 + x^2)} - \dfrac{\arctan x}{x^2}, & \text{if} \quad x \neq 0 \\ 0, & \text{if} \quad x = 0 \end{cases}$$

Solution Exercise 03

2) we know, $\arctan : \mathbb{R} \to]-\frac{\pi}{2}, \frac{\pi}{2}[$ is continuous on $\mathbb{R}$ then over $[0, x]$, $x > 0$ and differentiable over $]0, x[$, according to mean value theorem there exists $c \in]0, x[$ such that

$$\arctan'(c) = \lim_{x \to 0} \frac{\arctan x - \arctan 0}{x - 0} = \frac{\arctan x}{x} = \frac{1}{1 + c^2}$$

other words,

$0 < c < x \Rightarrow 1 < 1 + c^2 < 1 + x^2 \Rightarrow \dfrac{1}{1 + x^2} < 1 + c^2 < 1,$

hence, $\dfrac{1}{1 + x^2} < \dfrac{\arctan x}{x}$

$$\text{then,} \quad \frac{x}{1 + x^2} < \arctan x.$$

Exercise 04

Prove the following inequalities :

1. $\forall x,\ y \in \mathbb{R}, |\arctan x - \arctan y| \leq |x - y|$
2. $\forall x \geq 0,\ x \leq e^x - 1 \leq xe^x$

Solution. exercise 04

1. According to the inequality of Main value theorem i.e (f differentiable function on D. if $\forall x, y \in D : |f'(x)| \leq M$:) then

$$|f(x) - f(y)| \leq |x - y|$$

The function arctan is differentiable on $\mathbb{R}$ and its derivative : $\arctan' x = \dfrac{1}{1+x^2}$, and $|\dfrac{1}{1+x^2}| \leq 1$, then

$$\forall x, y \in \mathbb{R} : |\arctan x - \arctan y| \leq |x - y|$$

2. We apply the Main value theorem on $[0, x]$ we obtain $e^x - e^0 = e^c(x - 0) \Leftrightarrow e^x - 1 = x.e^c$, otherwise we have

$$0 \leq c \leq x \Leftrightarrow x \leq xe^c \leq xe^x \Leftrightarrow x \leq e^x - 1 \leq xe^x$$

End Chapter 05