Real Functions of Real variable

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Concept of function

To specify a function f you must

1. Give a rule which tells you how to compute the value $f(x)$ of the function for a given real number x, and,
2. Say for which real numbers x the rule may be applied.

. The set of numbers for which a function is defined is called its **domain**. The set of all possible numbers $f(x)$ as x runs over the domain is called the **range or image** of the function.

Definition

A real function f is function from a subset D of $\mathbb{R}$ to $\mathbb{R}$

Example

Definition domain of the function $f : x \mapsto \dfrac{2}{\sqrt{2-x}}$, is $D =]-\infty, 2[$.

Remark

The set $C = \{x \in \mathbb{D} : (x, f(x))\}$ *is called the* ***graph*** *of the function* f.

In mathematics, an **elementary function** is a function of a single variable composed of particular simple functions

Example

Basic examples

- Polynomial functions :
  $x \mapsto a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n \quad a_i \in \mathbb{R},\ i = 0, \dots, n$
- Trigonometric functions : $x \mapsto \sin x,\ \cos x,\ \tan x,\ \cot x.$
- Exponential function $x \mapsto \exp x$, or $\quad e^x$
- Logarithm function $x \mapsto \ln x$
- Inverse trigonometric functions :
  $x \mapsto \arcsin x, \arccos x, \arctan x, arccotx.$
- Hyperbolic functions :
  $$x \mapsto \sinh x = \frac{e^x - e^{-x}}{2}, \quad x \mapsto \cosh = \frac{e^x + e^{-x}}{2}, \text{ or}$$
  $$x \mapsto Shx = \frac{e^x - e^{-x}}{2}, \quad x \mapsto Ch = \frac{e^x + e^{-x}}{2},$$
- Inverse hyperbolic functions : $x \mapsto argshx, \quad argchx$

Definition

Let f and g two functions defined on $D \subseteq \mathbb{R}$

1. **The sum** : $f+g : D \to \mathbb{R}$, $(f+g)(x) = f(x)+g(x)$,
2. **The product** : $f.g : D \to \mathbb{R}$, $(f.g)(x) = f(x).g(x)$,
3. **The multiplies** : $\lambda.g : D \to \mathbb{R}$, $(\lambda.f)(x) = \lambda.f(x)$,

Definition

Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ tow functions. Then :

- $f \geq g$ if $\quad \forall x \in D \quad f(x) \geq g(x)$;
- $f \geq 0$ if $\quad \forall x \in D \quad f(x) \geq 0$;
- $f > 0$ if $\quad \forall x \in D \quad f(x) > 0$;
- f constant on D if $\exists\, a \in \mathbb{R}/\ \forall x \in D : \quad f(x) = a$;
- f is zero on D if $\forall\, x \in D : \quad f(x) = 0$;

Definition

Let $f : D \to \mathbb{R}$ is function. We say that :

- f is bounded from above over D if $\exists\, M \in \mathbb{R}\ /\forall x \in D : \quad f(x) \leq M$;
- f is bounded from below over D if $\exists\, m \in \mathbb{R}\ /\forall x \in D : \quad f(x) \geq m$;
- f is bounded over D If $\exists\, M \in \mathbb{R}\ /\forall x \in D : \quad |f(x)| \leq M$.

Definition

Let $f : D \to \mathbb{R}$ is function. We say that :

- f is increasing on D if $\forall x, y \in D,\ x \leq y \Rightarrow f(x) \leq f(y)$;
- f is strictly increasing on D if $\forall x, y \in D,\ x < y \Rightarrow f(x) < f(y)$;
- f is decreasing on D si $\forall x, y \in D,\ x \leq y \Rightarrow f(x) \geq f(y)$;
- f is strictly decreasing on D if $\forall x, y \in D,\ x < y \Rightarrow f(x) > f(y)$;
- f is monotone (resp. strictly monotone) on D if f is increasing or decreasing (resp. strictly increasing or strictly decreasing) on D.

For example the function $f : x \mapsto \sqrt{x}$ is increasing

$$\forall x_1,\ x_2 \in [0, +\infty[: x_1 \leq x_2 \Rightarrow \sqrt{x_1} \leq \sqrt{x_2}.$$

Remark

Let f function defined on I in the form $I = [-a, a]$ or $I =]-a, a[$

- *f is even function if $\forall x \in I : f(-x) = f(x)$,*
- *f is odd function if $\forall x \in I : f(-x) = -f(x)$,*

Example

f function defined on symmetric interval, such that $f = f_1 + f_2$

- Prove that the function $f_1(x) = \dfrac{f(x) + f(-x)}{2}$ is even.
- Prove that the function $f_2(x) = \dfrac{f(x) - f(-x)}{2}$ is odd.
- What can we conclude.

1. For tow even functions, what can be say about, the sum, the product and the compound (composed).
2. the same question if the tow functions is odd.
3. Also if one is even and second is odd.

Solution :

1.) Let f and g two even functions

$$(f+g)(-x) = f(-x)+g(-x) = f(x)+g(x) = (f+g)(x) \quad (f+g \text{ is even})$$

$$(f.g)(-x) = f(-x).g(-x) = f(x).g(x) = (f.g)(x) \quad (f.g \text{ is even})$$

$$(f \circ g)(-x) = f(g(-x)) = f(g(x)) = (f \circ g)(x) \quad (f \circ g \text{ is even})$$

Solution

2.) Let f and g two odd functions

$$(f+g)(-x) = f(-x)+g(-x) = -f(x)-g(x) = -(f+g)(x) \quad (f+g \text{ is odd})$$

$$(f.g)(-x) = f(-x).g(-x) = -f(x).(-g(x)) = (f.g)(x) \quad (f.g \text{ is even})$$

$$(f \circ g)(-x) = f(g(-x)) = f(-g(x)) = (f \circ g)(x) \quad (-f \circ g \text{ is odd})$$

3.) We assume that f is even and g is odd

$$(f+g)(-x) = f(-x)+g(-x) = f(x)-g(x) = (f-g)(x) \quad (f+g \ ?)$$

$$(f.g)(-x) = f(-x).g(-x) = f(x).(-g(x)) = -(f.g)(x) \quad (f.g \text{ is odd})$$

$$(f \circ g)(-x) = f(g(-x)) = f(-g(x)) = (f \circ g)(x) \quad (f \circ g \text{ is even})$$

Definition

Let $f : \mathbb{R} \to \mathbb{R}$ is a function, and T is a real number, $T > 0$. The function f is called periodic if $\forall x \in \mathbb{R}, \quad f(x+T) = f(x)$
The smallest such T (if it exists) is called the fundamental period of the function.

Example

- sin and cos are periodic functions of period 2π.
- tan and coth are periodic functions of period π.

Exercise : The function $f(x) = x - [x]$

$$f(x+1) = x + 1 - [x+1] = x + 1 - [x] - 1 = x - [x]$$

f is periodic function of period $T = 1$.

Let f real function defined by :

$$f(x) = \ln(|\sin(\frac{\pi}{2}x)|).$$

Find the domain of f ? the function f is even ? odd ? periodic ?
f defined if $|\sin(\frac{\pi}{2}x)| \neq 0$

$$\sin(\frac{\pi}{2}x) = 0 \Leftrightarrow \frac{\pi}{2}x = k\pi \Leftrightarrow x = 2k,\ k \in \mathbb{Z}$$

$\diamond$ so f is defined for all real, unless the even integers. Hence $D = \mathbb{R} - \{2\mathbb{Z}\}$.
$\diamond$ We have $\forall x \in D,\ -x \in D$, so

$$f(-x) = \ln(|\sin(\frac{\pi}{2}(-x))|) = \ln(|\sin(\frac{\pi}{2}x)|) = f(x)$$

then f is even.

Exercise

◇ We have

$$f(x+2) = \ln(|\sin(\frac{\pi}{2}(x+2))|) = \ln(|\sin(\frac{\pi}{2}x+\pi)|)$$

$$= \ln(|-\sin(\frac{\pi}{2}x)|) = \ln(|\sin(\frac{\pi}{2}x)|) = f(x).$$

f is periodic of period $T = 2$.

Limit

The notion of a limit is a fundamental concept of calculus. More particularly, limits allow us to look at what happens in a *very, very* small region around a point.

Example

Values of the real function f defined by $f(x) = \dfrac{x^2 - 1}{x - 1}$ my be computed near to $x_0 = 1$

x	0.9	0.99	0.999	$0.9999 \rightarrow$	$\leftarrow 1.0001$	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	$1.99999 \rightarrow$	$\leftarrow 2.0001$	2.001	2.01	2.1

We deduce that $\lim\limits_{x \to 1} f(x) = 2$.

Remark

A small region around a point x_0 is called **neighbourhood**.

Definition

For $x_0 \in \mathbb{R}$, an open interval as the form $]x_0 - \delta, x_0 + \delta[$ for some $\delta > 0$ is called neighbourhood

Definition

Let $f : I \to \mathbb{R}$ is a real function. Let $x_0 \in \mathbb{R}$ is point of I
Let $l \in \mathbb{R}$. We say that f for a limit l at x_0 if

$$\forall \epsilon > 0, \quad \exists \delta > 0, \quad \forall x \in I : |x - x_0| < \delta \Rightarrow |f(x) - l| < \epsilon.$$

Or we can write

$$\forall \epsilon > 0, \quad \exists \delta > 0, \quad \forall x \in I : x_0 - \delta < x < x_0 + \delta \Rightarrow l - \epsilon < f(x) < l + \epsilon.$$

it's also said that $f(x)$ tends to l as x tends to x_0. We write :

$$\lim_{x \to x_0} f(x) = l.$$

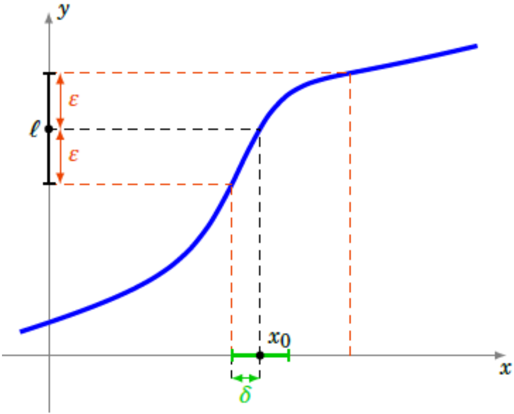


Figure 1 – Limit function

Example

Let f function defined on $\mathbb{R} - \{0\}$. We use the definition to limit prove that $\lim_{x\to 0}(x \sin \frac{1}{x}) = 0$

$$\forall \epsilon > 0, \quad \exists \delta > 0, \quad \forall x \in \mathbb{R} - \{0\} : \ |x - 0| < \delta \Rightarrow |x \sin \frac{1}{x} - 0| < \epsilon.$$

$$|x \sin \frac{1}{x} - 0| < \epsilon \Rightarrow |x|| \sin \frac{1}{x}| < |x|\epsilon$$

It's enough to choose $\delta = \epsilon$.

Example

A same question from $\lim_{x\to 4}(2x - 1) = 7$

$$\forall \epsilon > 0, \quad \exists \delta > 0, \quad \forall x \in \mathbb{R} : \ |x - 4| < \delta \Rightarrow |2x - 1 - 7| < \epsilon.$$

$$|2x - 8| = 2|x - 4| < \epsilon \Rightarrow |x - 4| < \frac{\epsilon}{\delta}$$

choose $\delta = \frac{\epsilon}{\delta}$.

• We say that the function f defined in a neighborhood of x_0 (except perhaps of x_0) has an $+\infty$ or $-\infty$ limit at x_0 if

$$\forall A > 0,\ \exists \delta > 0, \quad : |x - x_0| < \delta \Rightarrow f(x) > A \Leftrightarrow \lim_{x \to x_0} f(x) = +\infty$$

$$\forall A > 0,\ \exists \delta > 0, \quad : |x - x_0| < \delta \Rightarrow f(x) < -A \Leftrightarrow \lim_{x \to x_0} f(x) = -\infty$$

• If f is defined in a neighborfood of ∞ and f has a limit L

$$\forall \epsilon > 0,\ \exists B > 0 : \quad x > B \Rightarrow |f(x) - L| < \epsilon \Leftrightarrow \lim_{x \to +\infty} f(x) = L$$

$$\forall \epsilon > 0,\ \exists B > 0 : \quad x < -B \Rightarrow |f(x) - L| < \epsilon \Leftrightarrow \lim_{x \to -\infty} f(x) = L$$

- That the function f has limit ∞ when $x \to \infty$

$$\forall A > 0,\ \exists B > 0, \quad : x > B \Rightarrow f(x) > A \Leftrightarrow \lim_{x \to +\infty} f(x) = +\infty$$

$$\forall A > 0,\ \exists B > 0, \quad : x < -B \Rightarrow f(x) < -A \Leftrightarrow \lim_{x \to -\infty} f(x) = -\infty$$

$$\forall A > 0,\ \exists B > 0, \quad : x > B \Rightarrow f(x) < -A \Leftrightarrow \lim_{x \to +\infty} f(x) = -\infty$$

$$\forall A > 0,\ \exists B > 0, \quad : x < -B \Rightarrow f(x) > A \Leftrightarrow \lim_{x \to -\infty} f(x) = +\infty$$

Operations on the limit

f	g	$f+g$	$f.g$	$\frac{f}{g}$
l	l'	$l+l'$	$l.l'$	$\frac{l}{l'}(l' \neq 0)$
l	∞	∞	$\infty(l \neq 0)$	0
∞	l'	∞	$\infty(l' \neq 0)$	∞
0	0	0	0	?
0	∞	∞	?	0
∞	0	∞	?	0
$+\infty$	$+\infty$	$+\infty$	$+\infty$	?
$-\infty$	$-\infty$	$-\infty$	$+\infty$	?
$+\infty$	$-\infty$	?	$-\infty$	?
$-\infty$	$+\infty$	?	$-\infty$	?

In the cases indicated by the question marks, there is indeterminacy.

Limits and inequalities

- If there is some neighbourhood of a

$$f(x) \leq g(x),$$

  and the limits of f and g exist at a, then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x).$$

- If the limits of f and g exist at a, and

$$\lim_{x \to a} f(x) < \lim_{x \to a} g(x),$$

  then in some neighbourhood of a

$$f(x) < g(x).$$

Theorem

(Squeeze theorem) If in some neighborhood of a $f(x) < g(x) < h(x)$, *and the limits of* f *and* h *exist at* a, *and*

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x),$$

then the limit of g *also exists at* a, *and*

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \lim_{x \to a} h(x)$$

Theorem

(0 times bounded is 0.) If $\lim_{x \to a} f(x) = 0$, *and* $g(x)$ *is bounded, then*

$$\lim_{x \to a} f(x)g(x) = 0.$$

Exercise

1.) Find the limit, when $x \to 1$, of the function

$$f(x) = \frac{\sqrt{x^2-1}+\sqrt{x}-1}{\sqrt{x-1}}$$

On the domain $]1, +\infty[$

$$\lim_{x\to 1^+}(\sqrt{x+1}+\frac{\sqrt{x}-1}{\sqrt{x-1}}) = \lim_{x\to 1^+}(\sqrt{x+1}+\frac{\sqrt{x-1}}{\sqrt{x}+1}) = \sqrt{2}$$

2.) Find the limit, when $x \to \pm\infty$, of the function

$$f(x) = \frac{\sqrt{x^2+1}-x+2}{x+3}$$

Exercise

- For $x > 0$, we can write

$$\lim_{x \to +\infty} \left(\frac{\sqrt{1 + \frac{1}{x^2}} - 1 + \frac{2}{x}}{1 + \frac{3}{x}} \right) = 0$$

- For $x < 0$

$$\lim_{x \to -\infty} \left(\frac{-\sqrt{1 + \frac{1}{x^2}} - 1 + \frac{2}{x}}{1 + \frac{3}{x}} \right) = -2$$

3.) The function f defined by

$$f(x) = \begin{cases} x + \frac{\sqrt{x^2}}{x}, & x \neq 0 \\ 1 & x = 0 \end{cases}$$

does it have a limit as x tends towards zero.

$$\lim_{x \to 0^+} (f(x)) = 1, \quad \lim_{x \to 0^-} (f(x)) = -1 \quad \text{Does not limit.}$$

Exercise

Let the function f defined by $f(x) = \dfrac{\sqrt{1+x} - (1+ax)}{x^2}$

- Find the finite limit b when x tend to zero.
- Deduce that there exist a function ϵ such that

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + x^2\epsilon(x) \quad \text{with } \lim_{x\to 0}(\epsilon(x)) = 0.$$

Solution

$\circ$ $f(x) = \dfrac{[\sqrt{1+x} - (1+ax)][\sqrt{1+x} + (1+ax)]}{x^2[\sqrt{1+x} + (1+ax)]} = \dfrac{(1+x) - (1+ax)^2}{x^2[\sqrt{1+x} + (1+ax)]}$

$= \dfrac{-a^2x^2 + (1-2a)x}{x^2[\sqrt{1+x} + (1+ax)]} = \dfrac{-a^2}{\sqrt{1+x} + (1+ax)} + \dfrac{1-2a}{x(\sqrt{1+x} + (1+ax))}$

In this latter form, it immediately appears that f does not finite limit b, when x tends to zero if we have $1-2a=0$ then $a=\frac{1}{2}$ end in this cases

$$\lim_{x\to 0} f(x) = b = -\frac{a^2}{2} = -\frac{1}{8}.$$

$\circ$ Find the ϵ function

We deduce about the precedent compute that

$$\lim_{x\to 0} \frac{\sqrt{1+x} - (1+\frac{x}{2})}{x^2} = -\frac{1}{8}$$

Solution

The function ϵ defined by

$$\epsilon(x) = \frac{\sqrt{1+x} - (1+\frac{x}{2})}{x^2} + \frac{1}{8},$$

such that $\lim_{x\to 0}(\epsilon(x)) = 0$ then

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + x^2\epsilon(x).$$

Definition

The function f is continuous at a if and only if there exists the limit of the function at a and the limit is $f(a)$. Or

$$\lim_{x \to a} f(x) = f(a)$$

In logical symbolism,

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in D_f : |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon.$$

For example.

$\lim_{x \to 4} \frac{1}{x} = \frac{1}{4}$ by definition we can choose $\delta = 20\epsilon$ so that f continuous at 4.

$\circ f$ is continuous on D_f, if its continuous at every point in D_f.

$\circ$ If f is *not continuous* at a we say that f is *discontinuous* at a or that a is a point of discontinuity of f.

Remark

- *Polynomials function is continuous on* $\mathbb{R}$
- *Constant function is continuous on the interval.*
- *Functions* $\cos, \sin$ *are continuous on* $\mathbb{R}$
- $\ln$ *is continuous on* $]0, +\infty[$
- *Exponential function is continuous on* $\mathbb{R}$

- The integer part function is not continuous at every point $a \in \mathbb{Z}$ because does not limit at these points.
- But is $a \in \mathbb{R}/\mathbb{Z}$ is continuous at a.

Corollary

- *A function f is continuous from right at a if and only if there exists its* **right-hand side limit** *at a and it is $f(a)$.*
- *A function f is continuous from left at a if and only if there exists its* **left-hand side limit** *at a and it is $f(a)$.*

For example : **signum** function is not continuous at 0

$$\lim_{x \to 0^+} sgn(x) = 1, \quad \lim_{x \to 0^+} sgn(x) = -1.$$

Example

The function f defined by

$$f(x) = \begin{cases} x + \frac{\sqrt{x^2}}{x}, & x \neq 0 \\ 1 & x = 0 \end{cases}$$

Study the continuity of f at zero.

Continuous

$$\lim_{x \to 0^+} (f(x)) = 1 = f(0),$$

f is continuous in the right.

$$\lim_{x \to 0^-} (f(x)) = -1 \neq f(0).$$

f is not continuous in the left.
Then

$$\lim_{x \to 0^+} (f(x)) \neq \lim_{x \to 0^-} (f(x)).$$

f is not continuous at 0.

Continuous

Definition

A function f is uniformly continuous on a set D if for every $\epsilon > 0$, there exists $\delta > 0$, such that $|f(x_1) - f(x_2)| < \epsilon$ for all $x_1, x_2 \in D_f$, and $|x_1 - x_2| < \delta$.

More briefly

$$\forall \epsilon > 0, \exists \delta > 0, \forall x_1, x_2 \in D_f : |x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \epsilon$$

Example

$f : x \mapsto x^2$ uniformly continuous on $[0, 1]$.

In fact

$$|x_1^2 - x_2^2| = |x_1 + x_2||x_1 - x_2| \Rightarrow |x_1 - x_2| < \frac{\epsilon}{2} \quad \text{choosing} : \delta = \frac{\epsilon}{2}. \ f\text{U.C}$$

Let f is function defined as follow $D - x_0 \to \mathbb{R}$
We say that f is continuity by extension, if there exist g defined by

$$g(x) = \begin{cases} f(x) & if\ x \neq x_0, \\ \lim_{x \to x_0} f(x) = l & if\ x = x_0. \end{cases}$$

Then g is continuous at x_0.

Example

Find the constant k such that the function defined by

$$f(x) = \begin{cases} 3x + 2 & if\ x < 2 \\ x^2 + k & if\ x \geq 2. \end{cases}$$

is continuous

Continuous

Theorem

Weierstrass theorem : *If a function is continuous in a bounded and closed interval, then the function has maximum and minimum value.*

Theorem

Intermediate value theorem : If the function $f(x)$ is continuous in the bounded and closed $[a, b]$ interval, then every value y between $f(a)$ and $f(b)$ is attained c in $[a, b]$, such that $y = f(c)$

In logical symbolism this theorem has the following expression :

$$f \in \mathcal{C}([a, b]), \text{ and } f(a).f(b) < 0 \Rightarrow \exists c \in]a, b[, \text{ such that } f(c) = 0.$$

Example

Let $f : [0, 1] \to [0, 1]$ continuous function. Prove that f has at least fixed point.

Example :Intermediate Value Theorem

Put for example $g(x) = f(x) - x$
g is continuous because the difference of two continuous functions ($x \mapsto x$, and f). Also

$$g(0) = f(0) - 0 = f(0) > 0, \quad g(1) = f(1) - 1 < 0$$

then $g(0).g(1) < 0$, according to the I.V.T there exist $c \in]0, 1[$ such that

$$g(c) = 0 \Rightarrow f(c) - c = 0 \Rightarrow f(c) = c.$$

Therefore f has a fixed point.

Exercise

f continuous function on $[a, b]$ such that $f(a) < 0$. Show that there exist $c \in]a, b[$ so that

$$f(c) = \frac{a - c}{b - c}.$$

we can choose

$$g(x) = (b-x)f(x) - (a-x)$$

g is continuous because product $(x \mapsto (b-x)f(x))$ and difference $(x \mapsto a-x)$ continuous functions, also

$$g(a) = (b-a)f(a) < 0$$

$$g(b) = (b-b)f(b) - (a-b) = b-a > 0.$$

according to the M.V.T there exist $c \in]a, b[$ such that

$$g(c) = 0 \Leftrightarrow (b-c)f(c) - (a-c) = 0 \Leftrightarrow f(c) = \frac{a-c}{b-c}$$

Definition

Let f be a function which is defined on some interval I and a be some number in the interval. The derivative of the function f at a is the value of the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

- f is said to be differentiable at a if this limit exists.
- f is called differentiable on the interval I, if it is differentiable at every point a in I

Other notation : one can substitute $x - a = h$ then

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$$

• For example, we went to compute the derivative function of the function f defined by $f(x) = x^2$ at any point $x_0 \in \mathbb{R}$

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h\to 0} \frac{(x_0+h)^2 - x_0^2}{h} = 2x_0$$

Then the derivative function is $f'(x) = 2x$

• For $f(x) = \cos x$ the derivative function is

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h\to 0} \frac{\cos(x_0+h) - \cos(x_0)}{h}$$

$$\lim_{h\to 0} \frac{-2\sin(\frac{x_0+h+x_0}{2}).\sin(\frac{x_0+h-x_0}{2})}{h} = \lim_{h\to 0} \frac{-\sin(\frac{2x_0+h}{2}).\sin(\frac{h}{2})}{\frac{h}{2}}$$

Then the derivative function is $\cos'(x) = -\sin(x)$

The function f has a tangent line at point a if and only if f is differentiable at a. The equation of the tangent line is

$$y = f'(a)(x - a) + f(a)$$

• If f is differentiable at a, then the function is continuous at a.
The converse of the theorem is not true
For example, $f(x) = |x|$ is continuous at 0, but not differentiable at 0.

$$\lim_{x \to 0^+} \frac{x - 0}{x} = 1 = f'_>(0) \neq f'_<(0) = -1.$$

Then f is not differentiable at 0.

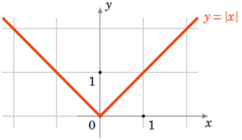


Figure 2 – Absolute value function

Remark

- *If* $\lim\limits_{x \to a^+} \dfrac{f(x) - f(a)}{x - a} = f'(a)$ *exists, this limit is the right derivative of* f *and equal* $f'_>(a)$.
- *If* $\lim\limits_{x \to a^-} \dfrac{f(x) - f(a)}{x - a} = f'(a)$ *exists, this limit is the left derivative of* f *and equal* $f'_<(a)$.
- *We say that* f *is derivative at* a *if*

$$f'_>(a) = f'_<(a).$$

Exercise

Let f function differentiable at a

1. Prove that
$$\lim_{x\to a} \frac{xf(a) - af(x)}{x-a} = f(a) - af'(a).$$
2. Application. Compute
$$\lim_{x\to a} \frac{x\cos(a) - a\cos(x)}{x-a}$$

Solution :

1) $xf(a)-af(x) = xf(a)-af(x)+af(a)-af(a) = f(a)(x-a)-a(f(x)-f(a))$

$$\frac{xf(a) - af(x)}{x-a} = \frac{f(a)(x-a) - a(f(x)-f(a))}{x-a} = f(a) - a\frac{f(x)-f(a)}{x-a}$$

then by passing of limit, we obtain

Exercise

$$\lim_{x\to a}\frac{xf(a)-af(x)}{x-a}=f(a)-a\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=f(a)-af'(a)$$

2) Put $f(x)\cos x$

$$\lim_{x\to a}\frac{x\cos(a)-a\cos(x)}{x-a}=\cos(a)+a\sin(a)$$

If f and g are differentiable at a, then

- for any $c \in \mathbb{R}$ f is differentiable at a, and
$$(c.f)'(a) = cf'(a)$$
- $f + g$ is differentiable at a, and
$$(f+g)'(a) = f'(a) + g'(a).$$
- $f.g$ is differentiable at a, and
$$(f.g)'(a) = f'(a).g(a) + f(a).g'(a)$$
- If $g(a) \neq 0$, then $\dfrac{f}{g}$ differentiable at a and
$$(\frac{f}{g})' = \frac{f'(a).g(a) - f(a).g'(a)}{g^2(x)}$$

Definition

Chain rule. If g is differentiable at a, and f is differentiable at $g(a)$, then $f \circ g$ is differentiable at a, and $(f \circ g)'(a) = g'(a)f'(g(a))$.

For example $gof(x) = \ln(1 + x^2)$

$$(gof)'(x) = (\ln(1 + x^2))' = (1 + x^2)' \ln'(1 + x^2) = 2x.(\frac{1}{1 + x^2}) = \frac{2x}{1 + x^2}.$$

Definition

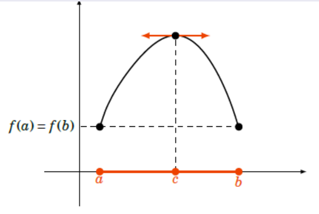
Derivative of the inverse function. If f is continuous and has an inverse in a neighborhood of the point a, and it is differentiable at a, then f^{-1} is differentiable at $f(a)$, and $(f^{-1})'(f(a)) = \dfrac{1}{f'(a)}$, such that $f'(a) \neq 0$.

Derivative of usual functions

Function	Derivative	Function $u(x)$	Derivative
x^n	$nx^{n-1} \quad n \in \mathbb{Z}$	u^n	$nu'u^{n-1} \quad n \in \mathbb{Z}$
$\frac{1}{x}$	$-\frac{1}{x^2}$	$\frac{1}{u}$	$-\frac{u'}{u^2}$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$	$\sqrt{u}$	$\frac{u'}{2\sqrt{u}}$
x^α	$\alpha x^{\alpha-1} \quad \alpha \in \mathbb{R}$	u^α	$\alpha u' u^{\alpha-1} \quad \alpha \in \mathbb{R}$
e^x	e^x	e^u	$u'e^u$
$\ln x$	$\frac{1}{x}$	$\ln u$	$-\frac{u'}{u}$
$\cos x$	$-\sin x$	$\cos u$	$-u'\sin u$
$\sin x$	$\cos x$	$\sin u$	$u'\cos u$
$\tan x$	$\frac{1}{\cos^2 x} = 1 + \tan^2 x$	$\tan u$	$\frac{u'}{\cos^2 u} = u'(1 + \tan^2 u)$

Theorem

Rolle's theorem. *If f is continuous on a closed interval $[a, b]$, and differentiable on the open interval $]a, b[$, and $f(a) = f(b)$, then there exists $c \in]a, b[$ such that $f'(c) = 0$.*



For example the function sin is continuous on $[0, \pi]$, differentiable on $]0, \pi[$, and $\sin(0) = \sin(\pi)$ then according to the Rolle's theorem there exist $c \in]0, \pi[$, such that

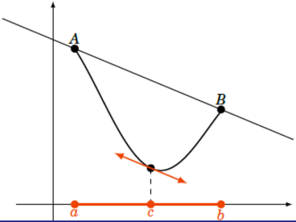
$$sin' c = 0 \Leftrightarrow \cos c = 0 \Leftrightarrow c = \frac{\pi}{2}.$$

Theorem

Mean value theorem. *If f is continuous on the closed interval $[a, b]$, and differentiable on the open interval $]a, b[$, then there exists a $c \in]a, b[$ such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Therefore, for any function that is continuous on $[a, b]$, and differentiable on $]a, b[$, there exists $c \in]a, b[$ such that the secant joining the endpoints of the interval $[a, b]$ is parallel to the tangent at c.



proof.
Let g function defined by

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a),$$

is continuous on $[a, b]$ because it's sum and product continuous functions $x \mapsto f(x)$ by data, $x \mapsto \frac{f(b)-f(a)}{b-a}$, constant function and $x \mapsto x - a$.
Also

$$g(a) = f(a), \quad g(b) = f(a),$$

so $g(a) = g(b)$. According to the Rolle's theorem there exist $c \in]a, b[$ such that $g'(c) = 0$.
Otherwise $g'(x) = f'(x) - \dfrac{f(b) - f(a)}{b - a}$ then

$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0 \Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

Relation between derivative and monotonic function

Corollary

Let f be continuous on $[a; b]$, and differentiable on $]a, b[$,.

- *f is monotonically increasing on $[a, b]$ if and only if for all $x \in]a, b[$, $f'(x) \geq 0$.*
- *If for all $x \in]a, b[$, $f'(x) > 0$ then $f(x)$ is strictly monotonically increasing on $[a, b]$.*
- *$f(x)$ is monotonically decreasing on $[a, b]$ if and only if for all $x \in]a, b[$, $f'(x) \leq 0$.*
- *If for all $x \in]a, b[$, $f'(x) < 0$ then $f(x)$ is strictly monotonically decreasing on $[a, b]$.*

For example if $f'(x) \geq 0$, we assume that $x, y \in]a, b[$ so that $x < y$ according to the M.V.T

$$f(y) - f(x) = f'(c)(y - x) \geq 0 \Rightarrow f(x) \leq f(x).$$

Then f is increasing.

Example

Use the Main Value Theorem, prove that for all $x > -1$,

$$\frac{x}{1+x} < \ln(1+x) < x$$

Let $]-1,+\infty[=]-1,0]\cup[0,+\infty[$ and the function

$$f : x \mapsto f(x) = \ln(1+x)$$

- Using the M.V.T on $]-1,0]$.0 we applied the M.V.T on $]x,0]$ so

$$f(0) - f(x) = f'(c)(0-x) \Rightarrow \ln(1+x) = \frac{x}{1+c}$$

$$x < c < 0 \Rightarrow 1+x < c+1 < 1 \Rightarrow 1 < \frac{1}{1+c} < \frac{1}{1+x}$$

then $\dfrac{x}{1+x} < \dfrac{x}{1+c} < x \Rightarrow \dfrac{x}{1+x} < \ln(1+x) < x.$

Corollary

Let f function differentiable on open interval I, if there exist $M > 0$ such that $|f'(x)| \leq M$ then for all x, y in I

$$|f(x) - f(y)| \leq M|x - y|$$

Example

For example the sin function, we have $|\sin'(x)| = |\cos(x)| \leq 1$, then

$$|\sin x - \sin y| \leq 1.|x - y|$$

- On $[0,+\infty[$ for $x>0$ we applied the M.V.T on $[0,x[$

$$f(x)-f(0)=f'(c)(x-0)\Rightarrow \ln x=\frac{x}{1+c}$$

Otherwise

$$0<c<x\Rightarrow 1<c+1<1+x\Rightarrow \frac{1}{1+x}<\frac{1}{1+c}<1$$

then $\dfrac{x}{1+x}<\dfrac{x}{1+c}<x\Rightarrow \dfrac{x}{1+x}<\ln(1+x)<x.$

Theorem

Let's assume that f and g are differentiable in a punctured neighborhood of a, f and g have limits at a, and either both limits are 0 or both limits are ∞, that is the limit of the quotient of the two function is critical. In this case if there exists the limit $\lim_{x\to a}\frac{f'(x)}{g'(x)}$, then also exists the limit $\lim_{x\to a}\frac{f(x)}{g(x)}$, and

$$\lim_{x\to a}\frac{f'(x)}{g'(x)}=\lim_{x\to a}\frac{f(x)}{g(x)}$$

This theorem is also valid for one-sided limits or limits at infinity or minus infinity.

Exercise 1 : Calculate the limits (Applying L'Hospital's theorem)

$$\lim_{x\to 0} \frac{\ln(1+x)-x}{x^2}, \quad \lim_{x\to 0} \frac{e^{x^2}-\cos x}{x^2}$$

$$\lim_{x\to 0} \frac{\ln(1+x)-x}{x^2} = \frac{0}{0}$$

$$\lim_{x\to 0} \frac{(\ln(1+x)-x)'}{(x^2)'} = \lim_{x\to 0} \frac{\frac{1}{1+x}-1}{2x} = \frac{0}{0}.$$

$$\lim_{x\to 0} \frac{(\frac{1}{1+x}-1)'}{(2x)'} = \lim_{x\to 0} \frac{\frac{-1}{(1+x)^2}}{2} = \frac{-1}{2}.$$

$$\lim_{x\to 0} \frac{e^{x^2}-\cos x}{x^2} = \lim_{x\to 0} \frac{2xe^{x^2}+\sin x}{2x} = \lim_{x\to 0} \frac{(2+4x^2)e^{x^2}+\cos x}{2} = \frac{3}{2}$$

Exercise 2 : Let f numerical function defined as follow

$$f(x) = \begin{cases} \dfrac{\ln(2-e^x)}{x}, & x < 0 \\ -1 - x, & x \geq 0. \end{cases}$$

- Study the continuity of f on $\mathbb{R}$.
- Study the differentiability of f on $\mathbb{R}$.
- f' is continuous on $\mathbb{R}$? (i.e $f \in \mathcal{C}^1(\mathbb{R})$)

Solution

For $x < 0$ we have $e^x < 1$, hence $2 - e^x > 1 > 0$, so $\ln(2 - e^x)$ is defined. Thus f is defined on all $\mathbb{R}$.

• Continuity

On $]-\infty, 0[$ and on $]0, \infty+[$ each branch is continuous. It remains to check continuity at $x = 0$.

Consider the left-hand limit :

$$\lim_{x \to 0^-} \frac{\ln(2 - e^x)}{x} = \frac{0}{0}.$$

$$\lim_{x \to 0^-} \frac{\ln(2 - e^x)}{x} = \lim_{x \to 0^-} \frac{(\ln(2 - e^x))'}{(x)'} = \lim_{x \to 0^-} \frac{-e^x}{2 - e^x} = -1.$$

The right-hand limit is

$$\lim_{x \to 0^+} (-1 - x) = -1,$$

and $f(0) = -1$. Hence f is continuous at 0, and therefore continuous on $\mathbb{R}$.

Solution

- Differentiability
- On the intervals $]-\infty, 0[$ and $]0, \infty[$.

◦ For $x < 0$ the numerator and denominator are differentiable and $x \neq 0$, so by the quotient rule

$$f'(x) = \frac{x \cdot (\ln(2 - e^x))' - \ln(2 - e^x)}{x^2} = \frac{x(-\frac{e^x}{2 - e^x}) - \ln(2 - e^x)}{x^2}, \quad x < 0.$$

- For $x > 0$ we have $f(x) = -1 - x$, hence $f'(x) = -1$ for $x > 0$.
- At $x = 0$.

Compute the right derivative :

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x} = \lim_{x \to 0^+} \frac{(-1 - x) - (-1)}{x} = \lim_{x \to 0^+} \frac{-x}{x} = -1.$$

Solution

For the left derivative consider

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x} = \lim_{x\to 0^-} \frac{\dfrac{\ln(2-e^x)}{x}+1}{x} = \lim_{x\to 0^-} \frac{\ln(2-e^x)+x}{x^2}.$$

The numerator and denominator both tend to 0, so apply L'Hôpital's theorem :

$$\lim_{x\to 0^-} \frac{(\ln(2-e^x)+x)'}{(x^2)'} = \lim_{x\to 0^-} \frac{\dfrac{-e^x}{2-e^x}+1}{2x}.$$

This new quotient is again of type $0/0$, so apply L'Hôpital once more :

$$\lim_{x\to 0^-} \frac{\dfrac{-e^x}{2-e^x}+1}{2x} = \lim_{x\to 0^-} \frac{\dfrac{d}{dx}\left(-\dfrac{e^x}{2-e^x}\right)}{2}.$$

Therefore

$$\lim_{x\to 0^-} \frac{\ln(2-e^x)+x}{x^2} = \lim_{x\to 0^-} \frac{-2e^x}{2(2-e^x)^2} = -1.$$

Solution

So the left derivative at 0 equals -1. Since both one-sided derivatives equal -1, f is differentiable at 0 and $f'(0) = -1$.
Thus

$$f'(x) = \begin{cases} \dfrac{x(-\dfrac{e^x}{2-e^x}) - \ln(2-e^x)}{x^2}, & x < 0, \\ -1, & x \geq 0. \end{cases}$$

$*$ Continuity of f'
Each branch of f' is continuous on its interval. We only check continuity at 0.
We already have $f'(0) = -1$ and $\lim_{x\to 0^+} f'(x) = -1$. For the left-hand limit consider

$$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} \frac{x(-\dfrac{e^x}{2-e^x}) - \ln(2-e^x)}{x^2}.$$

Definition

Let f be a differentiable function, and let f' be its derivative. The derivative of f' (if it has one) is written f'' and is called the second derivative of f. Similarly, the derivative of the second derivative, if it exists, is written f''' and is called the third derivative of f. Continuing this process, one can define, if it exists, the n-th derivative as the derivative of the (n-1)-th derivative. These repeated derivatives are called higher-order derivatives. The n-th derivative is also called the **derivative of order** n (or n-th-order derivative : first, second, third-order derivative, ...) and denoted $f^{(n)}$.

Example

The higher derivatives of the function $f : x \mapsto f(x) = \ln(1+x)$ are :

$$f'(x) = \frac{1}{1+x}, \quad f''(x) = -\frac{1}{(1+x)^2}, \quad f^{(3)}(x) = -\frac{1.2}{(1+x)^3},$$

$$f^{(4)}(x) = -\frac{-2.3}{(1+x)^4} \ldots, f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{(1+x)^n}$$

By induction demonstrate we prove that $f^{(n+1)}(x) = \dfrac{(-1)^{n+2} n!}{(1+x)^{n+1}}$ We have

$$f^{(n+1)}(x) = (f^n)' = \frac{(-1)^{n+1}(-1)(n-1)!n(1+x)^{n-1}}{((1+x)^n)^2}$$

$$= \boxed{\frac{(-1)^{n+2} n!}{(1+x)^{n+1}}}$$

Leibniz's rule

Generalizes the product rule (which is also known as **Leibniz's rule**). It f and g are n-times differentiable functions, then the product $f.g$ is also n-times differentiable and its n-th derivative given by

$$(f.g)^n = f^{(n)}.g + C_n^1 f^{(n-1)} g^{(1)} + ... + C_n^k f^{(n-k)} g^{(k)} + ... + f.g^{(n)}$$

$$= \sum_{k=0}^{k=n} C_n^k f^{(n-k)} g^{(k)}$$

Example

Using Leibnitz formula ; calculate the derivative $7-th$ of the function $h(x) = x^3. \ln x$

End of Chapter : Real Functions Of Real variable