Numerical Sequences

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What is a Sequence ?

A sequence is an infinite list (technically an infinite tuple (or set)) of numbers, for example :

$$2, 4, 6, 8, 10, \ldots$$

or

$$1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$$

or

$$1, 0, 1, 0, 1, 0, 1, 0, \ldots$$

An integral part of Analysis I is the study of various properties of real sequences and their relationships.

Example : Compound Interest

For example if we invest a sum S at an annual rate of 10%
S_n represents the sum that we will obtain after n years, we have

$$S_0 = S, \quad S_1 = S(1.1), \quad S_2 = S(1.1)^2, \ldots \quad S_n = S(1.1)^n$$

After $n = 10$ years :

$$S_{10} = S(1.1)^{10} \approx 2.59S$$

Definition

A function $u : \mathbb{N} \to \mathbb{R}$ whose domain of definition the set $\mathbb{N}$ is called a **sequence**.

The values $u(n) = u_n$ of the function u are called the terms of the sequence. In this connection the sequence itself is denoted (u_n), and also written as

u_1, the 1-st element
u_2, the 2-nd element
u_3, the 3-rd element
u_3, the 4-th element
...,
u_n, the n-th element It is called a sequence of elements in $\mathbb{R}$.
The element u_n is called the n-th term (or general term)of sequence.

title

- A numerical sequence (u_n) is **strictly increasing** if $u_n < u_{n+1}, \quad \forall n \in \mathbb{N}$
- A numerical sequence (u_n) is **increasing** if $u_n \leq u_{n+1}, \quad \forall n \in \mathbb{N}$
- A numerical sequence (u_n) is **strictly decreasing** if $u_n > u_{n+1}, \quad \forall n \in \mathbb{N}$
- A numerical sequence (u_n) is **decreasing** if $u_n \geq u_{n+1}, \quad \forall n \in \mathbb{N}$

For example the sequence (u_n) such that the n-th term $u_n = n^2$ est strictly increasing.

Example

In fact.

$$u_{n+1} - u_n = (n+1)^2 - n^2 = 2n+1 > 0.$$

Then (u_n) is strictly increasing for all $n \in \mathbb{N}$ The sequence (v_n) defined by $v_n = \dfrac{n}{n+1}$

$$u_{n+1} - u_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{1}{(n+1)(n+2)} > 0.$$

Remark

A numerical sequence is said to be **monotonic** *if it is increasing or decreasing.*

Remark

Let (u_n) a numerical sequence

- *If* $\frac{u_{n+1}}{u_n} \geq 1$ *is* **increasing** *and* **strictly increasing** *if*
  $\frac{u_{n+1}}{u_n} > 1, \quad \forall n \in \mathbb{N}$
- *If* $\frac{u_{n+1}}{u_n} \leq 1$ *is* **decreasing** *and* **strictly decreasing** *if*
  $\frac{u_{n+1}}{u_n} < 1, \quad \forall n \in \mathbb{N}$

For example $u_n = (n+1)(n+2)...(n+n)$

$$\frac{u_{n+1}}{u_n} = \frac{(n+2)(n+3)...(2n+1)(2n+2)}{(n+1)(n+2)...(n+n)} = 2(2n+1) > 1$$

strictly increasing.

Exercise

1. $u_n = a.n + (-1)^n$

$$u_{n+1} - u_n = a + 2(-1)^{n+1} = \begin{cases} a+2, n \text{ odd} \\ a-2, n \text{ even} \end{cases}$$

If $a \geq 2$ then $u_{n+1} - u_n$ is always positive and (u_n) is increasing.
If $0 \leq a < 2$ here the sequence (u_n) neither increasing nor decreasing.

2. $u_n = n^n - n!$

$$u_{n+1} - u_n = (n+1)^{n+1} - (n+1)! - n^n + n! = (n+1)^{n+1} + n!(1-n-1) - n^n$$
$$= (n+1)^{n+1} - n!n - n^n$$

we have $1.2.3...n \leq n.n.n...n$ so $n! \leq n^n$ hence

$$u_{n+1} - u_n \geq (n+1)^{n+1} - n^n(n+1) = (n+1)((n+1)^n - n^n) \geq 0.$$

Then (u_n) is increasing.

3. $u_n = n^2 - e^n$

We consider the function f defined on $\mathbb{R}^+$ as follow

$$f(x) = x^2 - e^x \text{ so } f(x) = 2x - e^x$$

f' attained its maximum at $x = \ln 2$, so $f(\ln 2) = 2\ln 2 - 2 < 0$ thus f' is negative then f is decreasing therefore (u_n) is also decreasing.

Convergence of a Sequence

Definition

If $\lim_{n\to\infty} u_n = l$, we say that (u_n) **converges** to l (or tends to l) and write :

$$u_n \to l \text{ as } n \to +\infty$$

A sequence that has a limit is convergent ; otherwise, it is divergent.

$$\lim_{n\to\infty} u_n = l \Leftrightarrow \forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n > N, |u_n - l| < \epsilon$$

We call the natural number N the *threshold* for the given ϵ.

Using the definition of limit, verify that

$$\lim_{n\to\infty} u_n = l \Leftrightarrow \forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n > N, |u_n - l| < \epsilon$$

1. $\lim_{n\to\infty} \frac{1}{n} = 0,$ since$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \epsilon,$ when $n > N = [\frac{1}{\epsilon}] + 1$ (We recall that $[\frac{1}{\epsilon}]$ is the integer part of the number $\frac{1}{\epsilon}$.)
2. $\lim_{n\to\infty} \frac{1+n}{n} = 1,$

$$\text{since } \left|\frac{1+n}{n} - 1\right| = \frac{1}{n} < \epsilon, \text{ when } n > N = [\frac{1}{\epsilon}] + 1$$

(4) $\lim_{n\to\infty} 1+\frac{(-1)^n}{n}=1,$ since

$$\left|1+\frac{(-1)^n}{n}-1\right|=\frac{1}{n}<\epsilon, \text{ when } n>N=[\frac{1}{\epsilon}]+1$$

(5) $\lim_{n\to\infty} \frac{\sin n}{n}=0,$ since

$$\left|\frac{\sin n}{n}-0\right|=\frac{1}{n}<\epsilon, \text{ when } n>N=[\frac{1}{\epsilon}]+1$$

(6) $\lim_{n\to\infty} \frac{1}{q^n}=0,$ if $|q|>1$ when $N\geq[\frac{\ln(1/\epsilon)}{\ln(q)}]+1$

Example

- The sequence
$$1, 2, \frac{1}{3}, 4, \frac{1}{5}, 6, \frac{1}{7}, \dots$$
whose n-th term is $u_n = n^{(-1)^n}, \ n \in \mathbb{N}$ is divergent.
- One can verify similarly that the sequence
$$1, -1, 1, -1, \dots,$$
for which $u_n = (-1)^n,$ has not limit.

Divergent Sequences

To Infinity

$$\forall A \in \mathbb{R}^+, \exists N \in \mathbb{N}, \forall n > N : u_n > A$$

Then we write $\lim u_n = +\infty$.

To Minus Infinity

$$\forall A \in \mathbb{R}^-, \exists N \in \mathbb{N}, \forall n > N : u_n < A$$

Then we write $\lim u_n = -\infty$.

- For example $\lim 2^n = +\infty$ in fact

$$\forall A \in \mathbb{R}, \exists N \in \mathbb{N}, \forall n > N : 2^n > A \Leftrightarrow n > \frac{\ln A}{\ln 2}, \quad n \geq \left[\frac{\ln A}{\ln 2}\right] + 1$$

Exercise

- Using the definition of limit show that the sequence $U_n = \dfrac{2n-3}{3n-1}$ is convergent to $\dfrac{2}{3}$.
- Let u_n sequence defined by the general term

$$u_n = \frac{2}{(2n+1)(2n+3)}$$

  - Write u_n as the form $u_n = \dfrac{\alpha}{2n+1} + \dfrac{\beta}{2n+3}$
  - Determine the monotonicity of the sequence.
  - Compute the limit of $S_n = u_0 + u_0 + \ldots + u_n$

Solution

1) **Show that the sequence** $U_n = \dfrac{2n-3}{3n-1}$ **converges to** $\dfrac{2}{3}$ **using the definition of limit.**
First compute the difference :

$$\left| U_n - \frac{2}{3} \right| = \left| \frac{2n-3}{3n-1} - \frac{2}{3} \right| = \left| \frac{3(2n-3) - 2(3n-1)}{3(3n-1)} \right| = \frac{7}{9n-3}.$$

Let $\varepsilon > 0$. We want $\dfrac{7}{9n-3} < \varepsilon$. This inequality is equivalent to

$$9n - 3 > \frac{7}{\varepsilon} \quad \Longleftrightarrow \quad n > \frac{7/\varepsilon + 3}{9}.$$

Choose

$$N = \left\lfloor \frac{7/\varepsilon + 3}{9} \right\rfloor + 1.$$

Then for every $n \geq N$ we have $\dfrac{7}{9n-3} < \varepsilon$, hence $\left|U_n - \dfrac{2}{3}\right| < \varepsilon$. By the definition of limit,

$$\lim_{n\to\infty} U_n = \frac{2}{3}.$$

2) Let the sequence u_n be defined by

$$u_n = \frac{2}{(2n+1)(2n+3)}, \qquad n \geq 0.$$

Solution

3) **Write** u_n **in the form** $\dfrac{\alpha}{2n+1} + \dfrac{\beta}{2n+3}$**.**
We find constants α, β such that

$$\frac{2}{(2n+1)(2n+3)} = \frac{\alpha}{2n+1} + \frac{\beta}{2n+3}.$$

Combine the right-hand side :

$$\frac{\alpha(2n+3) + \beta(2n+1)}{(2n+1)(2n+3)} = \frac{2}{(2n+1)(2n+3)}.$$

Thus for all n,

$$\alpha(2n+3) + \beta(2n+1) = 2.$$

Solution

By identification, we have

$$\begin{cases} 2\alpha + 2\beta = 0, \\ 3\alpha + \beta = 2. \end{cases}$$

From the first equation $\beta = -\alpha$. Substitute into the second :

$$3\alpha - \alpha = 2 \implies 2\alpha = 2 \implies \alpha = 1, \quad \beta = -1.$$

Hence

$$\boxed{u_n = \frac{1}{2n+1} - \frac{1}{2n+3}.}$$

Solution

4) **Determine the monotonicity of the sequence (u_n).** Each u_n is positive because the denominator is positive. To check monotonicity compute

$$u_n - u_{n+1} = \frac{2}{(2n+1)(2n+3)} - \frac{2}{(2n+3)(2n+5)} = \frac{8}{(2n+1)(2n+3)(2n+5)}$$

Since the denominator is positive for all $n \geq 0$, we have $u_n - u_{n+1} > 0$. Thus $u_{n+1} < u_n$ for all n, so (u_n) is strictly decreasing.
Also since $u_n > 0$ and decreasing, it converges and its limit must be 0 (one can also observe $\lim_{n\to\infty} u_n = 0$ directly).

Solution

5) **Compute the limit of the partial sums**

$$S_n = u_0 + u_1 + \cdots + u_n.$$

Using the decomposition found above, the sum telescopes :

$$\begin{aligned} S_n &= \sum_{k=0}^{n} \left(\frac{1}{2k+1} - \frac{1}{2k+3} \right) \\ &= \left(\frac{1}{1} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{2n+1} - \frac{1}{2n+3} \right). \end{aligned}$$

All intermediate terms cancel, leaving

$$S_n = 1 - \frac{1}{2n+3}.$$

Therefore

$$\boxed{\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1-\frac{1}{2n+3}\right) = 1.}$$

1. A sequence taking only one value is called a constant sequence.
2. If there exists a number a and an index p such that $u_n = a$ for all $n > p$, the sequence (u_n) will be called **ultimately constant.**
   $\star$ For example the sequence of n-th term

$$u_n = \max\{\frac{11}{10}, \frac{n+1}{n}\},$$

   for $n \geq 10$, (u_n) is **ultimately constant.** ✓ verify
3. The sequence (u_n) is **bounded** if there exists M such that $|u_n| < M$ for all $n \in \mathbb{N}$.

Properties

Theorem

a. *Every ultimately constant sequence is convergent.*
b. *A convergent sequence cannot have two distinct limits.*
c. *Every convergent sequence is bounded.*

Proof

a.) Let (u_n) be an ultimately constant sequence. Then there exists an integer p and a real number a such that

$$u_n = a, \quad \text{for all } n \geq p.$$

Let $\varepsilon > 0$. For all $n \geq p$, we have

$$|u_n - a| = 0 < \varepsilon.$$

Hence, by the definition of limit, $\lim\limits_{n \to \infty} u_n = a$. Therefore, every ultimately constant sequence is convergent.

Proof

b.) Suppose that a sequence (u_n) converges to two distinct limits l and l' with $l \neq l'$. Let $\varepsilon = \dfrac{|l - l'|}{3} > 0$. By definition of the limit, there exist integers N_1 and N_2 such that :

$$|u_n - l| < \varepsilon \quad \text{for all } n \geq N_1,$$

and

$$|u_n - l'| < \varepsilon \quad \text{for all } n \geq N_2.$$

Take $N = \max(N_1, N_2)$. Then for all $n \geq N$,

$$|l - l'| \leq |l - u_n| + |u_n - l'| < 2\varepsilon = \frac{2}{3}|l - l'|,$$

which is impossible since $|l - l'| > 0$. Therefore, a convergent sequence cannot have two distinct limits.

Proof

c.) Let (u_n) be a convergent sequence with $\lim_{n\to\infty} u_n = l$. By the definition of the limit, there exists an integer N such that

$$|u_n - l| < 1 \quad \text{for all } n \geq N.$$

This implies that $|u_n| \leq |l| + 1$ for all $n \geq N$. The finite set $\{u_1, u_2, \ldots, u_{N-1}\}$ is also bounded. Let

$$M = \max\Big(|l| + 1, |u_1|, |u_2|, \ldots, |u_{N-1}|\Big).$$

Then $|u_n| \leq M$ for all $n \in \mathbb{N}$, so the sequence is bounded.

Theorem

Let (a_n), *and* (b_n) *two convergent real sequences, with* $a_n \longrightarrow a$ *and* $b_n \longrightarrow b$. *Then we have*

1. *The sequence* $(a_n + b_n)$ *convergent with*

$$\lim_{n\to\infty}(a_n + b_n) = a + b \quad (\textit{sum rule}).$$

2. *The sequence* $(a_n.b_n)$ *convergent with*

$$\lim_{n\to\infty}(a_n.b_n) = a.b \quad (\textit{product rule}).$$

Theorem

1. *For $c \in \mathbb{R}$,the sequence $(c.a_n)$ convergent with*
$$\lim_{n\to\infty}(c.a_n) = c.a \quad (\textit{constant multiple rule}).$$
2. *If $b_n \neq 0$, for all $n \in \mathbb{N}$ and $b \neq 0$ Then the sequence $(\dfrac{a_n}{b_n})$ convergent with*
$$\lim_{n\to\infty}(\frac{a_n}{b_n}) = \frac{a}{b} \quad (\textit{quotient rule}).$$

Exercise

Find the limits of the following sequences

$$\lim_{n\to\infty}\frac{\sin(n^2+1)}{n^2+1},\quad \lim_{n\to\infty}\left(\frac{\sqrt{n+2}-\sqrt{n+1}}{\sqrt{n+1}-\sqrt{n}}\right),$$

$$\lim_{n\to\infty}\frac{\sin n}{n}(\sqrt{n+1}-\sqrt{n}),\quad \lim_{n\to\infty}(\sqrt{n^4+n^2}-n^2),$$

Solution

- $\lim_{n\to\infty} \dfrac{\sin(n^2+1)}{n^2+1}$

Since $|\sin(x)| \leq 1$ for all real x, we have

$$\left|\frac{\sin(n^2+1)}{n^2+1}\right| \leq \frac{1}{n^2+1}.$$

As $\dfrac{1}{n^2+1} \to 0$ when $n \to \infty$, by the squeeze theorem we obtain

$$\boxed{\lim_{n\to\infty} \frac{\sin(n^2+1)}{n^2+1} = 0.}$$

Solution

$$\lim_{n\to\infty} \frac{\sqrt{n+2}-\sqrt{n+1}}{\sqrt{n+1}-\sqrt{n}}$$

Rationalize each difference :

$$\sqrt{n+2}-\sqrt{n+1} = \frac{(n+2)-(n+1)}{\sqrt{n+2}+\sqrt{n+1}} = \frac{1}{\sqrt{n+2}+\sqrt{n+1}},$$

$$\sqrt{n+1}-\sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}}.$$

Therefore,

$$\frac{\sqrt{n+2}-\sqrt{n+1}}{\sqrt{n+1}-\sqrt{n}} = \frac{\frac{1}{\sqrt{n+2}+\sqrt{n+1}}}{\frac{1}{\sqrt{n+1}+\sqrt{n}}} = \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+2}+\sqrt{n+1}}.$$

Solution

Divide numerator and denominator by $\sqrt{n}$:

$$\frac{\sqrt{1+\frac{1}{n}}+1}{\sqrt{1+\frac{2}{n}}+\sqrt{1+\frac{1}{n}}} \longrightarrow \frac{1+1}{1+1}=1.$$

Hence,

$$\boxed{\lim_{n\to\infty} \frac{\sqrt{n+2}-\sqrt{n+1}}{\sqrt{n+1}-\sqrt{n}}=1.}$$

Solution

$$\lim_{n\to\infty} \frac{\sin n}{n}(\sqrt{n+1} - \sqrt{n})$$

We know that

$$\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \leq \frac{1}{2\sqrt{n}}.$$

Then

$$\left|\frac{\sin n}{n}(\sqrt{n+1} - \sqrt{n})\right| \leq \frac{1}{n} \cdot \frac{1}{\sqrt{n+1} + \sqrt{n}} \leq \frac{1}{2n^{3/2}}.$$

Since $\frac{1}{2n^{3/2}} \to 0$, by the squeeze theorem we obtain

$$\boxed{\lim_{n\to\infty} \frac{\sin n}{n}(\sqrt{n+1} - \sqrt{n}) = 0.}$$

Solution

$\lim_{n\to\infty}(\sqrt{n^4+n^2}-n^2)$

Factor out n^2 under the square root :

$$\sqrt{n^4+n^2}-n^2 = n^2\left(\sqrt{1+\tfrac{1}{n^2}}-1\right).$$

Multiply by the conjugate :

$$n^2\left(\sqrt{1+\tfrac{1}{n^2}}-1\right) = n^2 \cdot \frac{\frac{1}{n^2}}{\sqrt{1+\frac{1}{n^2}}+1} = \frac{1}{\sqrt{1+\frac{1}{n^2}}+1}.$$

As $n\to\infty$, $\sqrt{1+\frac{1}{n^2}}\to 1$, so

$$\boxed{\lim_{n\to\infty}(\sqrt{n^4+n^2}-n^2) = \frac{1}{2}.}$$

Theorem

- *Let* (a_n), *and* (b_n) *two convergent real sequences, with* $a_n \longrightarrow a$ *and* $b_n \longrightarrow b$*. If* $a < b$*, then there exists an index* $N \in \mathbb{N}$ *such that* $a_n < b_n$, *for all* $n > N$.
- *Suppose the sequences* $(a_n), (b_n)$*, and* (c_n) *such that* $a_n < b_n < c_n$ *for all* $n > N \in \mathbb{N}$*. the sequences* (a_n) *and* (c_n) *both converge to the same limit, then the sequence* (b_n) *also converges to that limit.*

Comparison theorem for sequences tending to $+\infty$)

For example $-1 \leq \sin(n^2+1) \leq 1$, then $\dfrac{-1}{n^2+1} \leq \dfrac{\sin(n^2+1)}{n^2+1} \leq \dfrac{1}{n^2+1}$

by passage to the limit $\lim\limits_{n\to\infty} \dfrac{\sin(n^2+1)}{n^2+1} = 0$.

• Let us investigate what happens if we consider a sequence (a_n) with $(an) \longrightarrow +\infty$and a sequences (b_n) which is controlled by (a_n) by $\forall n \in \mathbb{N} : a_n \leq b_n$

Theorem

(Comparison theorem for sequences tending to $+\infty$). Let (a_n) and (b_n) be real sequences. Suppose that $(an) \longrightarrow +\infty$ and that there exists an $N \in \mathbb{N}$ such that for all $n > N$, we have $b_n > a_n$. Then, $(b_n) \longrightarrow +\infty$

In general. Two sequences are adjacent if the first is increasing, the second is decreasing, and their difference converges to 0.

Definition

Two real sequences (u_n) and (v_n) are called adjacent if (u_n) is *increasing*, (v_n) is *decreasing*, and $\lim_{n\to\infty}(u_n - b_n) = 0$.

Example 2 : The two sequences (u_n) and (v_n) defined by

$$u_n = 1 + \frac{1}{n+1}, \quad v_n = 1 - \frac{1}{n+1}, \quad \forall n \in \mathbb{N}$$

are adjacent.

Solution

1. Monotonicity of u_n :

$$u_n = 1 + \frac{1}{n+1}$$

Since the function $n \mapsto \frac{1}{n+1}$ is strictly decreasing, it follows that :

$$u_{n+1} = 1 + \frac{1}{n+2} < 1 + \frac{1}{n+1} = u_n.$$

Thus, (u_n) is a **decreasing** sequence.

2. Monotonicity of v_n :

$$v_n = 1 - \frac{1}{n+1}$$

Since $\frac{1}{n+1}$ decreases, we get :

$$v_{n+1} = 1 - \frac{1}{n+2} > 1 - \frac{1}{n+1} = v_n.$$

Hence, (v_n) is an **increasing** sequence.

Solution

$$v_{n+1} = 1 - \frac{1}{n+2} > 1 - \frac{1}{n+1} = v_n.$$

Hence, (v_n) is an **increasing** sequence.

3. Order between u_n and v_n :

$$u_n - v_n = \left(1 + \frac{1}{n+1}\right) - \left(1 - \frac{1}{n+1}\right) = \frac{2}{n+1} > 0,$$

so $u_n > v_n$ for all n.

4. Limit of their difference :

$$u_n - v_n = \frac{2}{n+1} \quad \Rightarrow \quad \lim_{n\to\infty}(u_n - v_n) = 0.$$

Exercise

Let

$$u_n = \sum_{k=1}^{n} \frac{1}{k^2}, \quad v_n = u_n + \frac{2}{n+1}$$

Show that u_n, v_n are adjacent.

Solution

1. Monotonicity of (u_n). For every $n \geq 1$,

$$u_{n+1} - u_n = \frac{1}{(n+1)^2} > 0,$$

so (u_n) is strictly increasing.
2. Monotonicity of (v_n). Compute the difference

$$\begin{aligned} v_{n+1} - v_n &= \left(u_{n+1} + \tfrac{2}{n+2}\right) - \left(u_n + \tfrac{2}{n+1}\right) = \left(u_{n+1} - u_n\right) + \frac{2}{n+2} - \frac{2}{n+1} \\ &= \frac{1}{(n+1)^2} - \frac{2}{(n+1)(n+2)} = \frac{(n+2) - 2(n+1)}{(n+1)^2(n+2)} \\ &= -\frac{n}{(n+1)^2(n+2)} < 0 \quad \text{for all } n \geq 1. \end{aligned}$$

Hence (v_n) is strictly decreasing.

Solution

3. Order between the sequences.

Since $v_n = u_n + \dfrac{2}{n+1}$ and $\dfrac{2}{n+1} > 0$, we have

$$u_n < v_n \qquad \text{for all } n \geq 1.$$

4. Difference tends to 0**.**

$$v_n - u_n = \frac{2}{n+1} \xrightarrow[n\to\infty]{} 0.$$

Conclusion.

The sequence (u_n) is increasing, (v_n) is decreasing, $u_n \leq v_n$ for all n, and $v_n - u_n \to 0$. Therefore (u_n) and (v_n) are adjacent sequences. $\square$

Exercise (U_n) and (V_n) be two sequences such that :

$$\begin{cases} U_0 \leq V_0, \quad 0 < \beta < \alpha \\ U_n = \dfrac{\alpha U_{n-1} + \beta V_{n-1}}{\alpha + \beta} \\ V_n = \dfrac{\alpha V_{n-1} + \beta U_{n-1}}{\alpha + \beta} \end{cases}$$

1. Let $W_n = U_n - V_n$. Prove that (W_n) is geometric sequence. Identify the common ratio q and the first term W_0.
2. Prove that (U_n) is increasing and that (V_n) is decreasing.
3. Deduce that (U_n) and (V_n) are adjacent sequences.
4. Find the limit l in function of U_0 and V_0.

Solution

1.) Show that (W_n) is a geometric sequence.

Let $W_n = U_n - V_n$. Then

$$\begin{aligned} W_n &= U_n - V_n \\ &= \frac{\alpha U_{n-1} + \beta V_{n-1}}{\alpha + \beta} - \frac{\alpha V_{n-1} + \beta U_{n-1}}{\alpha + \beta} \\ &= \frac{(\alpha - \beta)(U_{n-1} - V_{n-1})}{\alpha + \beta} \\ &= q\, W_{n-1}, \end{aligned}$$

where

$$q = \frac{\alpha - \beta}{\alpha + \beta}, \qquad W_0 = U_0 - V_0.$$

Hence $W_n = q^n W_0$. Since $0 < \beta < \alpha$, we have $0 < q < 1$.

2.) Show that (U_n) is increasing and (V_n) is decreasing.

We compute the successive differences :

$$U_n - U_{n-1} = \frac{\alpha U_{n-1} + \beta V_{n-1} - (\alpha+\beta) U_{n-1}}{\alpha+\beta} = \frac{\beta(V_{n-1} - U_{n-1})}{\alpha+\beta}$$
$$= -\frac{\beta}{\alpha+\beta} W_{n-1}.$$

Similarly,

$$V_n - V_{n-1} = \frac{\beta(U_{n-1} - V_{n-1})}{\alpha+\beta} = \frac{\beta}{\alpha+\beta} W_{n-1}.$$

Since $U_0 \leq V_0$, we have $W_0 \leq 0$. As $0 < q < 1$, $W_n = q^n W_0 \leq 0$ for all n. Therefore :

$$U_n - U_{n-1} = -\frac{\beta}{\alpha+\beta} W_{n-1} \geq 0, \quad V_n - V_{n-1} = \frac{\beta}{\alpha+\beta} W_{n-1} \leq 0.$$

Thus, (U_n) is increasing and (V_n) is decreasing.

Solution

Show that (U_n) and (V_n) are adjacent.

We already know $U_n \leq V_n$ for all n since $W_n = U_n - V_n \leq 0$. Moreover,

$$V_n - U_n = -W_n = -q^n W_0 \geq 0,$$

and because $0 < q < 1$, we have $V_n - U_n \to 0$ as $n \to \infty$. Hence, (U_n) and (V_n) are adjacent sequences (one increasing, the other decreasing, and their difference tends to 0).

Find the common limit in terms of U_0 and V_0.

Let $T_n = U_n + V_n$. Then

$$\begin{aligned} T_n &= U_n + V_n \\ &= \frac{\alpha U_{n-1} + \beta V_{n-1}}{\alpha+\beta} + \frac{\alpha V_{n-1} + \beta U_{n-1}}{\alpha+\beta} \\ &= \frac{(\alpha+\beta)(U_{n-1}+V_{n-1})}{\alpha+\beta} = T_{n-1}. \end{aligned}$$

So (T_n) is constant and $T_n = T_0 = U_0 + V_0$ for all n. Since (U_n) and (V_n) are adjacent, they converge to the same limit l. Taking the limit in the equality $T_n = U_n + V_n$, we obtain :

$$2l = U_0 + V_0 \quad \Rightarrow \quad \boxed{l = \frac{U_0+V_0}{2}}.$$

Recursive sequence

Definition

Let $f : \mathbb{R} \to \mathbb{R}$ a real function. A recursively-defined sequence is an relation that expresses each element of a sequence as a function of the preceding one.

$$u_0 \in \mathbb{R}, \quad u_{n+1} = f(u_n)$$

Then a recurrence sequence is defined by two data u_0 the first term and recurrence relation

$$u_0, \quad u_1 = f(u_0), \quad u_2 = f(u_1) = fof(u_0), \dots \quad , u_n = fof \dots of(u_0)$$

Examples

Example 1 : Let $u_0 = 2,$ and for $n \geq 0, \quad u_{n+1} = f(u_n)$

$$u_1 = 1 + \sqrt{2},\ u_2 == fof(2) = 1 + \sqrt{1 + \sqrt{2}},$$

$$, u_3 = fofof(2) = 1 + \sqrt{1 + \sqrt{1 + \sqrt{2}}}, \ldots$$

Example 2 :(Fibonacci sequence) is defined as follow : The succeeding terms are dependent on the last two preceding terms

$$F_0 = 0,\ F_1 = 1,\ \text{and for } n \geq 2, \quad F_n = F_{n-1} + F_{n-2}$$

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, \ldots$$

Examples

Two simple examples of recursive definitions are for **arithmetic sequences and geometric sequences.** An arithmetic sequence has a common difference, or a constant difference between each term.

$$u_n - u_{n-1} = r \quad \text{or} \quad u_n = u_{n-1} + r$$

A geometric sequence has a common ratio

$$u_n = q.u_{n-1}, \quad or \quad \frac{u_n}{u_{n-1}} = q$$

Again, in this case it is relatively easy to find a formula for the n-th term : $u_n = u_0.q^n$

Fixed Point Concept

Definition

A fixed point of a function f is a point x so that

$$f(x) = x.$$

Proposition

If f is a function and the recursive sequence converge to l, then l is a solution of the equation $f(l) = l$

In fact

$$l = \lim_{n\to\infty} u_{n+1} = \lim_{n\to\infty} (f(u_n)) = f(\lim_{n\to\infty} u_n) = f(l)$$

Example

Let f recursive sequence defined by

$\Big\{ \quad 1$

Exercise

Let u_n) and let the sequence be defined by

$$\begin{cases} u_0 \in \mathbb{R}, \\ u_{n+1} = u_n - u_n^2. \end{cases}$$

1. Prove that there exist n_0, such that (u_n) is stationery (or ultimately constant).
2. Prove that if $u_0 \in [0, 1]$, (u_n) is monotone and bounded.
3. deduce that (u_n) is convergent and find its limit.

Solution

1.) **Stationary (Ultimately Constant) Sequence**

• If for some integer k, $u_k = 0$, then

$$u_{k+1} = u_k - u_k^2 = 0,$$

and so $u_n = 0$ for all $n \geq k$. Hence, the sequence becomes constant from that point onward.

• If for some k, $u_k = 1$, then

$$u_{k+1} = 1 - 1^2 = 0,$$

and thus, for all $n \geq k+1$, $u_n = 0$. Therefore, the sequence is ultimately constant if and only if some term u_k is equal to 0 or 1.

Conversely, suppose that the sequence is ultimately constant, i.e. there exists N and $c \in \mathbb{R}$ such that $u_n = c$ for all $n \geq N$. Then substituting into the recurrence relation gives :

$$c = c - c^2 \quad \Rightarrow \quad c^2 = 0 \quad \Rightarrow \quad c = 0.$$

Thus, the only possible stationary value is 0.

Solution

2. Monotonicity and Boundedness for $u_0 \in [0,1]$

Assume $u_0 \in [0,1]$. We show by induction that $u_n \in [0,1]$ for all n.
$u_0 \in [0,1]$ is true by assumption.
Suppose $0 \leq u_n \leq 1$. Then :

$$u_{n+1} = u_n(1 - u_n) \geq 0,$$

and

$$u_{n+1} = u_n(1 - u_n) \leq u_n.$$

Hence $0 \leq u_{n+1} \leq u_n \leq 1$. Thus (u_n) is nonincreasing and all terms remain in $[0,1]$.

3. Convergence and Limit

Since (u_n) is bounded below by 0 and monotone decreasing, it converges. Let $\ell = \lim_{n\to\infty} u_n$. Passing to the limit in the recurrence relation gives :

$$\ell = \ell - \ell^2 \quad \Rightarrow \quad \ell^2 = 0 \quad \Rightarrow \quad \ell = 0.$$

Therefore,

$$\boxed{\lim_{n\to\infty} u_n = 0.}$$

Cauchy Criterion

We now have a test that allows us to establish that a monotonic sequence converges without knowing its limit.

Definition

A sequences (u_n) is called fundamental or Cauchy sequence if for any $\epsilon > 0$ there exists an index $N \in \mathbb{N}$ such that $|u_m - u_n| < \epsilon$ whenever $n > N$, and $m > N$ i.e

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \backslash \forall m, n \geq N : |u_m - u_n| < \epsilon.$$

For example, the sequence $(\frac{1}{n})_{n \in \mathbb{N}^*}$ is a Cauchy sequence.

$$|u_m - u_n| = |\frac{1}{m} - \frac{1}{n}| = |\frac{n-m}{mn}|$$

So, if $m, n \geq N$, then $mn \geq N^2$ hence

$$|\frac{n-m}{mn}| \leq \frac{|n-m|}{N^2}.$$

The maximum possible difference (when $|n-m|=1$) so

$$|u_m - u_n| = |\frac{n-m}{mn}| \leq \frac{|n-m|}{N^2} \leq \frac{1}{N^2}$$

In generally

$$|u_m - u_n| \leq \frac{1}{N}$$

Given $\epsilon > 0$, choose $N > \dfrac{1}{\epsilon}$ then for all $m, n \geq N : |u_m - u_n| < \epsilon$. Hence (u_n) satisfies the Cauchy criterion. Then (u_n) is Cauchy sequence.

Theorem

A numerical sequence converges if and only if is a Cauchy sequence.

For example the sequence $(-1)^n$ is not Cauchy sequence, because has not limit. Even though this fact is obvious we shall give a formal verification. The negation of the statement that (u_n) is Cauchy sequence is the following :

$$\exists \epsilon > 0, \forall N \in \mathbb{N}, \exists m, n \geq N : |u_m - u_n| \geq \epsilon.$$

That is, there exists $\exists \epsilon > 0$, such that for any $N \in \mathbb{N}$, two numbers n, m larger than N for which $|u_m - u_n| \geq \epsilon$. In our cases it suffices to set $\epsilon = 1$ then for any $N \in \mathbb{N}$, we shall have
$|u_{N+1} - u_{N+2}| = |1 - (-1)| = 2 > 1 = \epsilon$

Other example that the sequence (u_n) defined by the n-th term

$$u_n = 1 + \frac{1}{2} + ... + \frac{1}{n},$$

is not Cauchy sequence, because if we take $m = 2n$

$$|u_m - u_n| = |u_{2n} - u_n| = \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{n+n} \geq \frac{1}{2n} + \frac{1}{2n} + ... + \frac{1}{2n} = \frac{1}{2}$$

So, for $\epsilon = \frac{1}{2}$, $\forall N \in \mathbb{N}, \exists n = N, \; m = 2N$ such that

$$|u_m - u_n| = |u_{2N} - u_N| \geq \frac{1}{2}$$

For all $n \in \mathbb{N}$, Cauchy criterion implies immediately that this sequence does not limit.
The sequence is increasing, therefore $\lim_{n\to\infty} u_n = +\infty$

Exercise

Let the sequence defined by the n-term

$$u_n = \sum_{k=1}^{n} \frac{1}{n+k}, \quad n \in \mathbb{N}^*$$

- Prove that (u_n) is increasing.
- $\forall n \in \mathbb{N} : \frac{1}{2} \leq u_n \leq 1.$

Exercise

1.) Show that $u_n = \sum_{k=1}^{n} \frac{1}{k^2}$ is Cauchy sequence

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \backslash \forall m > n \geq N : |u_m - u_n| < \epsilon.$$

$$|u_m - u_n| = |\sum_{k=1}^{m} \frac{1}{k^2} - \sum_{k=1}^{n} \frac{1}{k^2}| = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \dots \frac{1}{m^2}$$

We have

$$\frac{1}{k^2} = \frac{1}{k.k} < \frac{1}{(k-1)k} = \frac{1}{k-1} - \frac{1}{k}$$

So

$$\frac{1}{(n+1)^2} = \frac{1}{(n+1)(n+1)} < \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$\frac{1}{(n+2)^2} = \frac{1}{(n+2)(n+2)} < \frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}$$

$$\frac{1}{(n+3)^2} = \frac{1}{(n+3)(n+3)} < \frac{1}{(n+2)(n+3)} = \frac{1}{n+2} - \frac{1}{n+3}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{1}{(m-1)^2} = \frac{1}{(m-1).(m-1)} < \frac{1}{(m-2)(m-1)} = \frac{1}{m-2} - \frac{1}{m-1}$$

$$\frac{1}{m^2} = \frac{1}{m.m} < \frac{1}{(m-1)m} = \frac{1}{m-1} - \frac{1}{m}$$

Hence

$$|u_m - u_n| < \frac{1}{n} - \frac{1}{m} < \frac{1}{N} < \epsilon$$

Given $\epsilon > 0$, choose $N = [\frac{1}{\epsilon}] + 1$ then for all $m, n \geq N : |u_m - u_n| < \epsilon$. Hence (u_n) satisfies the Cauchy criterion. Then (u_n) is Cauchy sequence.

Show that $u_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$ and $u_n = \sum_{k=1}^{n} \frac{3}{2\ln k}$ are not Cauchy sequences.

End of Chapter : Numerical Sequences