Field of Complex Numbers

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Outline

Complex numbers

Problem: The equation

$$x^2 = -4,$$

does not solution in $\mathbb{R}$, that's to say the field of real numbers has a obvious flaw. that's reason to extend the set of real number a an other set, where the previous equation does a solution.
Idea: We will imagine that there exists a non-real number i such that $i^2 = -1$ then

$$x^2 = i^2 4 \Leftrightarrow x = 2i \notin \mathbb{R} \text{ and } x = -2i \notin \mathbb{R}$$

which admits two different roots.

Construction the set of complex numbers

We consider the set $\mathbb{C}$ defined by

$$\mathbb{C} = \{(a,b) : a,\ b \in \mathbb{R}\},$$

provided by two laws: for all $(a,b) \in \mathbb{C},\ (c,d) \in \mathbb{C}$

$$(a,b) + (c,d) = (a+c, b+d), \quad (a,b).(c,d) = (ac - bd, ad + bc).$$

So $(\mathbb{C}, +, .)$ its commutative filed, because

1. $+$ is commutative, associative, $(0,0)$ addition identity and $(-a,-b)$ the inverse element to (a,b), and
2. . is commutative, associative and the multiplication distributive over the addition. (1.0) is multiplication identity. the inverse element of (a,b) is $(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2})$

For $(a,b) \in \mathbb{C}: \ (a,b) = (a,0) + (0,1)(b,0)$,
we accept that there is an identification between $(a,0) \in \mathbb{C} \equiv a \in \mathbb{R}$.
And we put $i = (0,1)$ so

$$i^2 = (0,1).(0,1) = (-1,0),$$

by identification we have: $(-1,0) \equiv -1$, then $i^2 = -1$ Therefore

$$(a,b) = (a,0) + (0,1)(b,0) = a + ib \in \mathbb{C}$$

Conjugate

Definition

If a, b are real and $z = a + ib$, then the complex number, $\overline{z} = a - ib$ is called the conjugate of z. a and$b$ are the real part and imaginary part of z, respectively.

we shall occasionally write $a = Re(z)$; $b = Im(z)$

Theorem

If z, and w are complex, then

1. $\overline{z + w} = \overline{z} + \overline{w}$
2. $\overline{z.w} = \overline{z}.\overline{w}$
3. $z + \overline{z} = 2Re(z); \quad z - \overline{z} = 2iIm(z)$
4. $-\overline{z} = \overline{-z}, \quad \overline{(\frac{z}{w})} = \frac{\overline{z}}{\overline{w}}(w \neq 0).$
5. *$z.\overline{z}$is real and positive.*

Module

Result:

- z is real $\Leftrightarrow \; z = \overline{z}$
- z is imaginary $\Leftrightarrow \; z = -\overline{z}$

Definition

If z is complex number its absolute value $|z|$ is non-negative square root of $z\overline{z}$ that is $|z| = \sqrt{z\overline{z}} = \sqrt{a^2 + b^2}$

Properties: If z, and w are complex, then

- $z = 0 \Leftrightarrow |z| = 0$
- $z\overline{z} = |z|^2$
- $|z.w| = |z|.|w|$
- $|Re(z)| \leq |z|; \quad |Im(z)| \leq |z|$

Exercise

1. Write under algebraic form the following complex numbers
   $z_1 = \dfrac{1}{2+3i}, \quad z_2 = \dfrac{1}{3-4i}$
2. Give the conjugate of $\dfrac{4-5i}{3+i}$.
3. Determine the location points $M(x, y)$ such that: $\dfrac{iz-1}{z-i}$ is real.

Solution

1. Compute the algebraic (rectangular) form of the complex numbers
$$z_1 = \frac{1}{2+3i}, \qquad z_2 = \frac{1}{3-4i}.$$
Rationalize each denominator:
$$z_1 = \frac{1}{2+3i} \cdot \frac{2-3i}{2-3i} = \frac{2-3i}{2^2+3^2} = \frac{2-3i}{13} = \frac{2}{13} - \frac{3}{13}i,$$
$$z_2 = \frac{1}{3-4i} \cdot \frac{3+4i}{3+4i} = \frac{3+4i}{3^2+4^2} = \frac{3+4i}{25} = \frac{3}{25} + \frac{4}{25}i.$$

2. Give the conjugate of $\dfrac{4-5i}{3+i}$.
First compute the quotient:
$$\frac{4-5i}{3+i} \cdot \frac{3-i}{3-i} = \frac{(4-5i)(3-i)}{3^2+1^2} = \frac{7-19i}{10} = \frac{7}{10} - \frac{19}{10}i.$$

Solution

Its complex conjugate is

$$\overline{\left(\frac{4-5i}{3+i}\right)} = \frac{7}{10} + \frac{19}{10}i.$$

3 Determine the points $M(x, y)$ (with $z = x + iy$) such that

$$\frac{iz-1}{z-i} \in \mathbb{R}.$$

Write $z = x + iy$. Then

$$iz - 1 = i(x+iy) - 1 = (-y-1) + ix, \qquad z - i = x + i(y-1).$$

Multiply numerator and denominator by the conjugate of the denominator:

$$\frac{iz-1}{z-i} = \frac{\big((-y-1)+ix\big)\big(x - i(y-1)\big)}{x^2 + (y-1)^2}.$$

Solution

The imaginary part of the numerator is

$$-(-y-1)(y-1) + x \cdot x = (y+1)(y-1) + x^2 = x^2 + y^2 - 1.$$

For the fraction to be real this imaginary part must vanish, hence

$$x^2 + y^2 - 1 = 0 \Longrightarrow x^2 + y^2 = 1.$$

We must also exclude the point where the denominator is zero, i.e. $z = i$ (which is $(x, y) = (0, 1)$), since the expression is undefined there.
Therefore the locus is the unit circle

$$x^2 + y^2 = 1,$$

with the point $(0, 1)$ omitted.

Representation of complex number

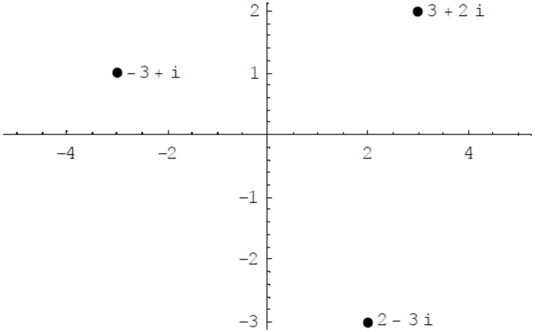
The complex numbers, having two components, their real and imaginary parts, can be represented as a plane; indeed, $\mathbb{C}$ is sometimes referred to as the complex plane.

The point (a, b) represents the complex number $a + bi$ so that

◦ the x-axis contains all the real numbers, and so is termed the real axis and

◦ the y-axis contains all those complex numbers which are purely imaginary (i.e. have no real part), and so is referred to as the imaginary axis.

Argument



Definition

We define θ to be the angle that the line connecting the origin to $z = x + iy,\ z \neq 0$ makes with the positive real axis, .The number θ is called **the argument** of z and is written $\arg z = \theta$.

Trigonometric form

We can write

$$z = r(\cos\theta + \sin\theta).$$

(polar or trigonometric form) The relations between z's Cartesian and polar co-ordinates are simple, we see that

$$x = r\cos\theta, \text{ and } \quad y = r\sin\theta, \quad r = \sqrt{x^2+y^2}, \quad \tan\theta = \frac{y}{x}.$$

Note that $\arg z$ is defined only up to multiples of 2π.
For example, the argument of $1+i$ could be $\dfrac{\pi}{4}$ or $\dfrac{9\pi}{4}$ or $\dfrac{-7\pi}{4}$ etc...
For simplicity, in this article we shall give all arguments in the range $0 \leq \theta < 2\pi$, so that $\dfrac{\pi}{4}$ would be the preferred choice here.

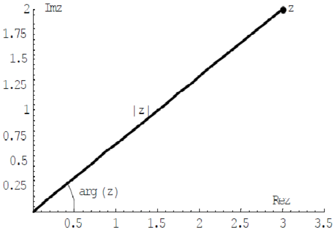


Figure 1: The argument

Remark

Note that the argument of 0 is undefined.

We can easy verified that

- $\arg(\overline{z}) = \arg(\frac{1}{z}) = \arg(z)[2\pi], \quad \arg(\frac{z}{w}) = \arg(z) - \arg(w)[2\pi]$
- $\arg(zw) = \arg(z) + \arg(w)[2\pi], \quad \arg(z^n) = n\arg(z)[2\pi]$

Examples

1. Give $\arg(i),\ \arg(1),\ \arg(-i),\ \arg(1+i)$.
2. Give the principal argument of $z=-2\sqrt{3}+2i,$ and $z'=3-4i$.
3. find the trigonometric form $z=-2(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5})$.

1) The principal arguments (in $(-\pi, \pi]$) :

$$\arg(i) = \frac{\pi}{2}, \qquad \arg(1) = 0, \qquad \arg(-i) = -\frac{\pi}{2}, \qquad \arg(1+i) = \frac{\pi}{4}.$$

2) for $z = -2\sqrt{3} + 2i$:

$$r = |z| = \sqrt{(-2\sqrt{3})^2 + 2^2} = \sqrt{12+4} = 4.$$

The argument principal (with $\operatorname{Im} z > 0$ and $\operatorname{Re} z < 0$) is

$$\arg(z) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}.$$

Thus the trigonometric form is

$$\boxed{z = 4\Big(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\Big)}.$$

For $z' = 3 - 4i$:

$$r' = |z'| = \sqrt{3^2 + (-4)^2} = 5,$$

and the principal argument (imaginary negative, real positive) is

$$\arg(z') = -\arctan\left(\frac{4}{3}\right) \quad \text{(en radians)} \approx -0.9273 \text{ (let } -53.13^\circ\text{)}.$$

Then the trigonometric form is

$$\boxed{z' = 5\Big(\cos\big(-\arctan(\tfrac{4}{3})\big) + i\sin\big(-\arctan(\tfrac{4}{3})\big)\Big)}.$$

3) We give the trigonometric form of

$$z = -2\Big(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\Big).$$

We remark that multiply by -1 adds π to the argument, hence

$$-2\Big(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\Big) = 2\Big(\cos\left(\tfrac{\pi}{5}+\pi\right) + i\sin\left(\tfrac{\pi}{5}+\pi\right)\Big)$$

$$= 2\Big(\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}\Big).$$

If we use $\cos\frac{\pi}{5} = \dfrac{1+\sqrt{5}}{4}$ and $\sin\frac{\pi}{5} = \dfrac{\sqrt{10-2\sqrt{5}}}{4}$:

$$z = -2\cos\frac{\pi}{5} - 2i\sin\frac{\pi}{5} = -\frac{1+\sqrt{5}}{2} - \frac{\sqrt{10-2\sqrt{5}}}{2}\, i.$$

Then:

$$\boxed{z = 2\Big(\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}\Big) \quad \text{et} \quad z = -\frac{1+\sqrt{5}}{2} - \frac{\sqrt{10-2\sqrt{5}}}{2}\, i\,.}$$

Exercise

Find the modulus and argument (polar form) of each following numbers.

$$z_1 = (1+\sqrt{3}i), \quad z_2 = (2+i)(3-i), \quad z_3 = (1+i)^5$$

Solution:

$$(1)\ z_1 = 1+\sqrt{3}i$$

$$|z_1| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2$$

$$\sin(\theta) = \frac{1}{2}, \quad \cos(\theta) = \frac{\sqrt{3}}{2},\ \arg(z_1) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$$

$$\boxed{z_1 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2e^{i\pi/3}}$$

Exercise

$$(2)\ z_2 = (2+i)(3-i)$$

First, expand:

$$z_2 = (2+i)(3-i) = 6 - 2i + 3i - i^2 = 7 + i$$

$$|z_2| = \sqrt{7^2 + 1^2} = \sqrt{50} = 5\sqrt{2}$$

$$\arg(z_2) = \tan^{-1}\left(\frac{1}{7}\right)$$

$$\boxed{z_2 = 5\sqrt{2}\left(\cos\left(\tan^{-1}\tfrac{1}{7}\right) + i\sin\left(\tan^{-1}\tfrac{1}{7}\right)\right) = 5\sqrt{2}\, e^{i\tan^{-1}(1/7)}}$$

Exercise

$$(3)\ z_3 = (1+i)^5$$

For $1+i$:

$$|1+i| = \sqrt{1^2+1^2} = \sqrt{2}, \qquad \arg(1+i) = \frac{\pi}{4}$$

Then

$$|z_3| = (\sqrt{2})^5 = 2^{5/2} = 4\sqrt{2}, \qquad \arg(z_3) = 5 \times \frac{\pi}{4} = \frac{5\pi}{4}$$

Hence,

$$\boxed{z_3 = 4\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = 4\sqrt{2}\, e^{i5\pi/4}}$$

Exponential form

Definition

(Euler's formula) For any real number θ, we denote $e^{i\theta}$ the complex number $\cos\alpha + i\sin\alpha$ is the modulus 1 and θ argument.

It follows from Euler's formula that, for any complex number z written in exponential form, $z = re^{i\theta}$

Theorem

For any $\theta_1,\ \theta_2 \in \mathbb{R}$, *we have*

- $e^{i\theta_1}.e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$
- $\dfrac{e^{i\theta_1}}{e^{i\theta_2}} = e^{i(\theta_1-\theta_2)}$
- $(e^{i\theta})^n = e^{in\theta} \quad n \in \mathbb{Z}$

Example

- Determine the exponential form for $z = \dfrac{(1+i)^4}{\sqrt{(3-i)^3}}$.
- Calculate $(1+i)^{14}$.

In fact

$$z = \frac{(\sqrt{2}e^{i\frac{\pi}{4}})^4}{(2e^{-i\frac{\pi}{6}})^3} = \frac{1}{2}e^{i\frac{3\pi}{2}} = -\frac{1}{2}.$$

$$(1+i)^{14} = -2^7$$

Theorem

$\forall \theta \in \mathbb{R}$, *we have*

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Proof.

$$e^{i\theta} + e^{-i\theta} = \cos\theta + i\sin\theta + \cos(-\theta) + i\sin(-\theta)$$

$$= \cos\theta + i\sin\theta + \cos\theta - i\sin\theta = 2\cos\theta$$

$$e^{i\theta} - e^{-i\theta} = \cos\theta + i\sin\theta - \cos(\theta) + i\sin(\theta) = 2i\sin\theta.$$

□

Example

Linear $\cos^4\theta$.

$$\cos^4\theta = (\frac{e^{i\theta}+e^{-i\theta}}{2})^4 = \frac{e^{4i\theta}+4e^{2i\theta}+6+4e^{-2i\theta}+e^{-4i\theta}}{16}$$

$$= \frac{\cos(4\theta)+4\cos(2\theta)+3}{8}$$

Same question from

$$\sin^3\theta\cos^2\theta = (\frac{e^{i\theta}-e^{-i\theta}}{2i})^3(\frac{e^{i\theta}+e^{-i\theta}}{2})^2$$

After calculation

$$= \frac{2\sin\theta+\sin 3\theta-\sin 5\theta}{16}.$$

Moiver's Formula

Theorem

For a real number θ and integer n we have that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

$$(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$$

Proof.

We have

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i\sin n\theta.$$

Therefore the first Moiver formula. The second formula is obtained we replace θ by $-\theta$. $\square$

This formula allows you to calculate $\cos n\theta$ and $\sin n\theta$ in function of $\cos\theta$ and $\sin\theta$.

The square root of a complex number

Let $w = \alpha + i\beta$, we want to find the square root of w, i.e find $z = x + iy$, such that $z^2 = w$, hence

$$\begin{cases} x^2 - y^2 = \alpha .....(1) \\ 2xy = \beta ....(2) \\ x^2 + y^2 = |w| ......(3) \end{cases}$$

The equations (1) and (3) allowed to find x and y and equation (2) allows us to remove the ambiguity on the signs.

Example

For example $w = \dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i$

$$\begin{cases} x^2 - y^2 = \dfrac{\sqrt{3}}{2}.....(1) \\ 2xy = \dfrac{1}{2}.......(2) \\ x^2 + y^2 = 1......(3) \end{cases}$$

(1)+(3): $x = \pm\sqrt{\dfrac{\sqrt{3}+2}{4}} = \pm\dfrac{\sqrt{2+\sqrt{3}}}{2}$

(3)-(1): $y = \pm\dfrac{\sqrt{2-\sqrt{3}}}{2}$

From (2): $xy > 0$, so x and y have the same sign. Therefore the roots

$$\frac{\sqrt{2+\sqrt{3}}}{2} + i\frac{\sqrt{2-\sqrt{3}}}{2}, \quad -\frac{\sqrt{2+\sqrt{3}}}{2} - i\frac{\sqrt{2-\sqrt{3}}}{2}$$

Example:Quadratic equation

We deal this type of equations with a simple numerical example, we consider

$$(1+i)z^2-(5+i)z+6+4i=0.$$

Then $\Delta = 16-30i$, so the square roots of Δ is $5+3i$ and $-5+3i$. Hence the solutions is

$$z_1 = 2-3i, \quad z_2 = 1+i$$

Exercise: Find the square roots of $-5-12i$, and hence solve the quadratic equation

$$z^2-(4+i)z+(5+5i)=0.$$

Exercise: Show that the complex number $1+i$ is a root of the cubic equation

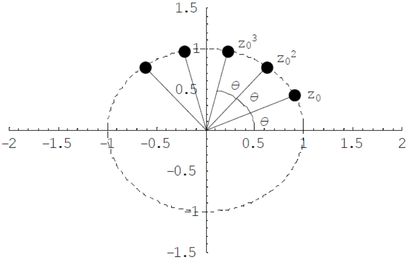
$$z^3+z^2+(5-7i)z-(10+2i)=0,$$

and hence find the other two roots.

Consider the complex number

$$z_0 = \cos\theta + i\sin\theta.$$

where $0 \leq \theta < 2\pi$. The modulus of z_0 is 1, and the argument of z_0 is θ.



Problem. Let n be a natural number. Find all those complex z such that $z^n = 1$.
We know from the Fundamental Theorem of Algebra that there are (counting repetitions) n solutions: these are known as the nth roots of unity.

Let's first solve $z^n = 1$ directly for $n = 2, 3, 4$.

- When $n = 2$ we have

$$0 = z^2 - 1 = (z-1)(z+1),$$

  and so the square roots of 1are ± 1.

- When $n = 3$ we can factories as follows

$$0 = z^3 - 1 = (z-1)(z^2+z+1).$$

  So the cube roots of 1 are $1,\ -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2},\ -\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}$

- When $n = 4$ we can factories as follows

$$0 = z^4 - 1 = (z^2-1)(z^2+1) = (z-1)(z+1)(z-i)(z+i),$$

  so that the fourth roots of 1 are $-1,\ i$ and $-i$.

Roots of unity (example)

Problem

Solve $z^n = 1$ for $z \in \mathbb{C}$.

- There are exactly n distinct solutions: $z_k = e^{2\pi ik/n}$ for $k = 0, 1, \ldots, n-1$.
- Special cases: $n = 2 : \{\pm 1\}$, $n = 3 : \{1, e^{\pm 2\pi i/3}\}$, $n = 4 : \{1, -1, i, -i\}$.

Exercise

1) Let α be a real number in the range $0 < \alpha < \frac{\pi}{2}$. Find the modulus and argument of the following numbers.

$$z_1 = \cos\alpha - i\sin\alpha, \quad z_2 = \sin\alpha - i\cos\alpha,$$

$$z_3 = 1 + i\tan\alpha, \quad z_4 = 1 + \cos\alpha + i\sin\alpha.$$

2) Calculate $\sin\frac{\pi}{5}, \cos\frac{\pi}{5}, \sin\frac{2\pi}{5}, \cos\frac{2\pi}{5}$

End chapter