Field of Real Numbers

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Outline

Representation of the set $\mathbb{R}$

Why study the real numbers $\mathbb{R}$. At the foundation of Analysis the real numbers.

Definition

It is admitted that there exists a set called the set of **natural numbers** $\mathbb{N}$ having the following properties:

- there exists a smallest element in $\mathbb{N}$, designed by 0.
- for all integers natural n there exists natural element $n^* = n + 1$ (n^* is called the next of n.)
- for all natural number n, $n^* \neq 0$.
- for all natural number n^*, m^* if $\quad n^* = m^*$ then $n = m$.
- **recurrence property:** let P be a property defined on $\mathbb{N}$, if $P(0)$ is verified (checked) and $P(n+1)$ then P is checked for all n natural

$$\mathbb{N} = \{0, 1, 2, \dots\}, \quad \mathbb{N}^* = \mathbb{N} \setminus \{0\}.$$

Induction demonstrate

Example

Prove that for all $n \in \mathbb{N} : 3^n - 1$ is even.
The preposition $P(n) : \forall n \in \mathbb{N} \exists k \in \mathbb{N} : 3^n - 1 = 2k$

Example

Prove that for all $n \in \mathbb{N} : 2^n \geq n$

Natural numbers $\mathbb{N}$

The set of natural numbers $\mathbb{N}$ has an obvious defect (or folaw), because if n and m two integers such that $n > m$ the algebraic equations

Problem (in $\mathbb{N}$)

$$n + x = m, \tag{1}$$

or

$$n.x = m \tag{2}$$

do not have solutions in $\mathbb{N}$.

Idea: we will extend $\mathbb{N}$ towards another set called the set of **whole numbers**. To remedy the situation (that to say the first equation has a solution).
The set of whole numbers, we note by $\mathbb{Z}$. The symbol $\mathbb{Z}$

From $\mathbb{N}$ to $\mathbb{Z}$ and $\mathbb{Q}$

- The set of whole numbers, also have a defect because to solve equations of the type (2) for every $m, n \in \mathbb{Z}$, we have to introduce another kind of numbers..
- Rationals $\mathbb{Q}$: fractions m/n with $m \in \mathbb{Z},\ n \in \mathbb{N}^*$.

Remark

There exists a rational number that can represent by decimal numbers

For example:

$$\frac{1}{5} = \frac{2}{10}, \quad \frac{4}{25} = \frac{16}{10^2}, \quad \frac{3}{500} = \frac{6}{10^3} \ldots,$$

The decimal number is rational number as the form

$$\frac{a}{10^n}, a \in \mathbb{Z}, n \in \mathbb{N}$$

Key properties of $\mathbb{Q}$

- Every non-zero integer n has multiplicative inverse $1/n \in \mathbb{Q}$.
- Rational numbers have repeating (eventually periodic) decimal expansions (or content infinity digits .)

For example:

$$\frac{1}{3} = 0.333333..\frac{1}{7} = 0.142857142857.., \frac{11}{7} = 1.5714285...,$$

$$\frac{11}{7} = 1.\underline{5714285}.$$

Question: Can represent a decimal number by fraction?
Answer: On condition, if the number of digits after decimal point is finite and periodic.

Example

$$A = 2.5454... = 2 + x, \text{ such that } \quad x = 0.5454...$$

we multiply the both sides $x = 0.5454...$ by 10^2

$$10^2.x = 10^2.0.5454...; \text{hence } 100x = 54 + x$$

$$x = \frac{54}{99} \text{ then } \quad A = 2 + \frac{54}{99} = \frac{252}{99}.$$

If $GCD(m, n) = r$ than it exists p and q such that $m = r.p$ $n = r.q$ and the solution is $x = \dfrac{p}{q}$ and $GCD(p, q) = 1$, x called rational number, and the set of rational numbers. This set is denoted by $\mathbb{Q}$

$$\mathbb{Q} = \{\frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{N}^*, \; GCD(a, b) = 1\}$$

In this set, one can add, subtract, multiply, and divide without any restrictions following the well known rules.

In this set, one can add, subtract, multiply, and divide without any restrictions.
Furthermore, one can solve linear algebraic equations of the form

$$ax + b = 0 \tag{3}$$

for arbitrary $a,\ b \in \mathbb{Q}$ with $a \neq 0$ uniquely by an $x \in \mathbb{Q}$.

Remark

1. *For all $n \in \mathbb{N}^*$ the equation $n.x = 1$, admits an only solution $x = \dfrac{1}{n}$.*
2. *For all $n \in \mathbb{N}^*$ the equation $n.x = -1$, admits an only solution $x = \dfrac{-1}{n}$.*
3. *Any rational number can be represented by a periodic decimal expansion.*

Problem: We can easily see that rational numbers have a defect, because the equation $x^2 = 2$ does not have a solution in $\mathbb{Q}$

Example

$$x^2 = 2, \tag{4}$$

We now show that the equation (4) is not satisfied by any rational x, we could write $x = \dfrac{p}{q}$ where p ad q are integers that are not both even.

Let we assume that is done. Then (4) implies

$$p^2 = 2q^2,$$

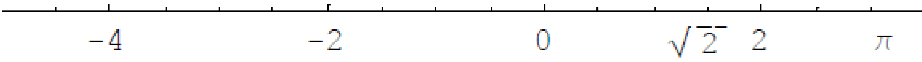
this show that p^2 is even. Hence p is even and so p^2 is divisible by 4, so that q^2 is even, which implies that q is even.

The assumption that (4) holds thus leads to the conclusion that both p and q are even, contrary to our choice of p and q. Hence (4) is impossible for rational x.

We extended the set of rational numbers into the set where does the equation (4) has solution. This set is the set of real numbers noted by $\mathbb{R}$ Let

$$\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}, \quad \mathbb{R}_+^* = \{x \in \mathbb{R} : x > 0\}$$

The same for $\mathbb{R}_-$ and $\mathbb{R}_+^*$.



Definition

The set of real numbers is the set of x-coordinate points on the line straight (O, i)

- The positive real numbers are the x-coordinate points on the right of O.
- The negative real numbers are the x-coordinate points on the left of O.

Remark

A no-rational number is said to be irrational and the set of these numbers is the irrational set denoted by $\mathbb{R}/\mathbb{Q}$.

For example: $\sqrt{2}$, e, π, are irrational numbers.

The algebraic structure of $\mathbb{R}$

We provided two laws on $\mathbb{R}$,

- **addition (+)**,
- **multiplication (.)**

for every a, b elements in $\mathbb{R}$ two new elements

- $a + b \in \mathbb{R}$,
- $a.b \in \mathbb{R}$

They are called the **sum** and the product of a, b. The operations addition and multiplication satisfy the following rules.

1. $(a+b)+c = a+(b+c)$ (Associativity)
2. $a+b = b+a$ (commutativity)
3. There is exactly one element in $\mathbb{R}$, called the zero and denoted by 0, such that
$$a + 0 = a \text{ for all } a \in \mathbb{R}$$
4. For all $a \in \mathbb{R}$ there exists exactly one $b \in \mathbb{R}$ such that $a+b=0$. The element b denoted by $-a$ and we will call it the negative to a.
5. $(ab)c = a(bc)$ (Associativity)
6. $ab = ba$ (commutativity)
7. There is exactly one element in $\mathbb{R} \setminus \{0\}$ called the **one** and denoted by 1, such that $a.1 = a$ for all $a \in \mathbb{R}$.
8. For every $a \in \mathbb{R} \setminus \{0\}$ there is exactly one element $b \in \mathbb{R}$, such that $ab = 1$. We denote b by a^{-1} or $\dfrac{1}{a}$ and we say a^{-1} is the inverse element of a.
9. $a(b+c) = ab + ac$. (Distributivity)

Notation

We set

- $a - b = a + (-b)$,
- $\dfrac{a}{b} = ab^{-1} = b^{-1}a$,
- $a - b$ the difference of a and b,
- $\dfrac{a}{b}$ the quotient of a and b.
- The operation $a, b \longmapsto a - b$ respectively $\dfrac{a}{b}$ subtraction and division.

Proposition

For all $a,\ b$ in $\mathbb{R}$

- $0.a = a.0 = 0.$
- $a.b = 0$ *if and only if* $a = 0$ *or* $b = 0.$
- $(-1).a = -a$

Order relation on $\mathbb{R}$

Between the elements of $\mathbb{R}$ there is a relation $\leq$, that is, for elements $x, y \in \mathbb{R}$. one can determine whether $x \leq y$ or not. Here the following conditions must hold:

Order properties

1. For all $x \in \mathbb{R}: \; x \leq x$ ($\leq$ reflexive). In other words, every element relates to itself.
2. For all $x, \; y \in \mathbb{R} : x \leq y$ and $y \leq x$ implies $x = y$ ($\leq$ anti-symmetric)
3. For all $x, \; y,$ and z if $x \leq y$ and $y \leq z$ implies that $x \leq z$ ($\leq$ transitive)
4. For all $x, \; y \in \mathbb{R} : x \leq y$ or $y \leq x$

A set on which there is a relation between pairs of elements satisfying axioms 1, 2, and 3, as you know, is said to be **partially ordered.** If in addition axiom 4 holds, that is, any two elements are **comparable or totally ordered**, the set is linearly ordered. Thus the set of real numbers is linearly ordered by the relation of inequality between elements.

Remark

Let $x,\ y \in \mathbb{R}$

- *If* $x \geq y$ *we can write* $y \leq x$
- *If* $x \geq y$ *and* $x \neq y$*we can write* $y < x$
- *If* $x \geq y$ *and* $x \neq y$*we can write* $y > x$

Integer part

Theorem

For all $x \in \mathbb{R}$ there exists a only one $k \in \mathbb{Z}$, such that $k \leq x < k+1$.

Definition

Let $x \in \mathbb{R}$, the relative integers k verify $k \leq x < k+1$, is said the **integer part** of x, denoted by $[x]$. Also

$$[x] \leq x < [x] + 1, \quad [x] = k$$

For example: $[1.5] = 1, \quad [0.95] = 0, \quad [-1.75] = -2, \quad \ldots$

Main properties

Remark

For all real x

$$[x] = \max\{k \in \mathbb{Z} : k \leq x < k+1\}$$

the greatest integer less or equal x

• Characterization: $\forall x \in \mathbb{R} : [x] \leq x < [x] + 1$.
In fact. By definition $[x]$ is the greatest integer $\leq x$. Hence $[x] \leq x$, but since the next integer: $[x] + 1$ is strictly greatest than $[x]$, we must have $x < [x] + 1$.

Main properties

• Monotonicity: If $x \leq y$ then $[x] \leq [y]$.
We have $[x] \leq x \leq y$ hence $[y] \leq y$. But $[y] \leq y$ is the greatest integer less or equal y. Then $[x] \leq [y]$.

Remark

We have $[x] \leq x < [x] + 1 \Leftrightarrow 0 \leq x - [x] < 1$.
$\{x\} = x - [x]$ *is the fractional part*

• Addition: $\forall x, y \in \mathbb{R} : [x] + [y] \leq [x + y]$.
We have $x = [x] + \{x\}$ and $y = [y] + \{y\}$ so
$x + y = [x] + [y] + \{x\} + \{y\}$ hence $[x] + [y] \leq x + y$ and $[x] + [y]$ is integer part of $x + y$ therefore $[x + y]$ is the greatest integer less or equal $x + y$. Then

$$[x] + [y] \leq [x + y]$$

Exercise

Solve in set of real numbers $\mathbb{R}$: $[\frac{x-1}{2}] = -2, \quad [\frac{1}{x}] = 3$

1. If $[a] = k$ then $k \leq a < k+1$. So

$$-2 \leq \frac{x-1}{2} < -1 \Leftrightarrow -3 \leq x < -1.$$

2. $$3 \leq \frac{1}{x} < 4 \Leftrightarrow \frac{1}{4} \leq x < \frac{1}{3}.$$

Exercise

Prove for all $n \in \mathbb{N}^*$: $\left[\frac{[nx]}{n}\right] = [x]$

We have $[nx] \leq nx \Rightarrow \frac{[nx]}{n} \leq x \Rightarrow \left[\frac{[nx]}{n}\right] \leq [x]$ .....(1) on otherwise,

$[x] \leq x \Rightarrow n[x] \leq nx \Rightarrow n[x] \leq [nx]$ hence $[x] \leq \frac{[nx]}{n}$ therefore

$[x] \leq \left[\frac{[nx]}{n}\right]$......(2)

From (1) and (2) we have the desired equality.

Exercise

Find the integer part of: $S = \dfrac{1}{\frac{1}{1980} + \frac{1}{1981} + ... + \frac{1}{2001}}$

We have

$$\begin{cases} \dfrac{1}{1980} \leq \dfrac{1}{1980} \\ \dfrac{1}{1981} \leq \dfrac{1}{1980} \\ . \\ . \\ \dfrac{1}{2001} \leq \dfrac{1}{1980} \end{cases} \Rightarrow \frac{1}{1980} + \frac{1}{1981} + ... + \frac{1}{2001} \leq 22.\frac{1}{1980}$$

So, $S \leq \dfrac{1980}{22} = 90.$

Also

$$\begin{cases} \dfrac{1}{1980} > \dfrac{1}{2001} \\ \dfrac{1}{1981} > \dfrac{1}{2001} \\ . \\ . \\ \dfrac{1}{2001} \geq \dfrac{1}{2001} \end{cases} \Rightarrow \frac{1}{1980} + \frac{1}{1981} + ... + \frac{1}{2001} > 22.\frac{1}{2001}$$

So $S < \dfrac{2001}{22} = 90 + \dfrac{21}{22}$

$$90 \leq S < 90 + \frac{21}{22} < 90 + 1 \Rightarrow [S] = 90.$$

supremum and infimum

Definition

Let $X \subset \mathbb{R}$ be nonempty set

- Is said to be **bounded from above or upper bound (Majorant)** if there exists a number $M \in \mathbb{R}$ such that

$$\forall x \in X, \ x \leq M.$$

1. M is the supremum of X exists, it is an upper bound of X and,
2. if M' another upper bound of X, than $M \leq M'$ (that's to say M is the **smallest upper bound** of X). If it is exists the supremum of X is denoted by $\sup(X)$

Definition

- Is said to be **bounded from below or lower bound (minorant)** if there exists a number $m \in \mathbb{R}$ such that

$$\forall x \in X, \ x \geq m.$$

1. m is the infimum of X exists, it is an lower bound of X and,
2. if m' another lower bound of X, than $m' \leq m$ (that's to say m is the **greatest lower bound** of X). If it is exists the infimum of X is denoted by $\inf(X)$.

Remark

- *If* $\inf(X) \in X$, *we call it* **minimum** *of* X, *and we denoted it by* $\min(X)$
- *If* $\sup(X) \in X$, *we call it* **maximum** *of* X, *and we denoted it by* $\max(X)$

For example, the set $X = \{x \in \mathbb{R} | 0 \leq x < 1\}$ has a minimal element. But, as one can easily verify, it has no maximal element.

Example

$$X = [1, 3], \sup(X) = \max(X) = 3, \quad \inf(X) = \min(X) = 1.$$

But if $X =]1, 3[, \sup = 3, \inf = 1.$ maximum and minimum does not exists.

Examples

Let

$$A = \{1, 2, 3, 4, 5\}$$

- Upper bounds: any number ≥ 5 like $(5, 6, 7...)$
- Smallest upper bound 5 so $\sup(A) = 5$.
- Lower bounds: any number ≤ 1 like $(..., -2, -1, 0)$
- Greatest lower bound 1 so $\inf(A) = 1$.

Here $\sup(A) = \max(A)$ and $\inf(A) = \min(A)$, since the set is bounded and finite.

$$B = \{\frac{1}{n}, \quad n \in \mathbb{N}^*\}$$

Upper bounds: clearly, 1 is an upper bound, and anything ≥ 1. Also the smallest upper bound $\sup(A) = 1$

Lower bound as $n \to \infty$, values gets closer to 0 but not reach it.

Greatest lower bound is $\inf(B) = 0$

Characterization of supremum and infimum

Theorem

1. *If a set* $X \subset \mathbb{R}$ *has supremum* $\sup(X) = M$ *if and only if*
   - M *is an upper bound of* X
   - *for all* $\epsilon > 0$*there exists* $a \in X$ *such that* $\sup(X) - \epsilon < a \leq \sup(X)$. *(leastness condition)*
2. *If a set* $X \subset \mathbb{R}$ *has infimum* $\inf(X) = m$ *if and only if*
   - m *is an lower bound of* X
   - *for all* $\epsilon > 0$*there exists* $a \in X$ *such that* $\inf(X) \leq a < \inf(X) + \epsilon$. *(greatness condition)*

In precedent example we can confirm that 0 is infimum of the set B

$$\forall \epsilon > 0, \exists a = \frac{1}{[\frac{1}{\epsilon}] + 1} : 0 \leq a < 0 + \epsilon$$

Exercise

Find $\sup(A)$ and $\inf(A)$ of

$$A = \{\sin \frac{2n\pi}{7}, \quad n\mathbb{Z}\}$$

We know that $\forall x \in \mathbb{R}, \sin x \in [-1, 1] \Rightarrow A \subset [-1, 1]$ so A is bounded and $\sup(A), \inf(A)$ exists.

Also the function $f : x \mapsto \sin \dfrac{2\pi x}{7}$ is periodic and their period is 7. Then

$$A = \{0, \sin \frac{2\pi}{7}, -\sin \frac{2\pi}{7}, \sin \frac{4\pi}{7}, -\sin \frac{4\pi}{7}, \sin \frac{6\pi}{7}, -\sin \frac{6\pi}{7}\}$$

$$\sup(A) = \sin \frac{4\pi}{7}, \inf(A) = -\sin \frac{4\pi}{7}.$$

Proposition

- *The supremum or infimum if there exists are alone*
- *If X no bounded from above, we write for convention* $\sup(X) = +\infty$
- *If X no bounded from below, we write for convention* $\inf(X) = -\infty$

Proposition

Let $A$and B two no-empty parts bounded in $\mathbb{R}$

1. *If $A \subset B$ then* $\sup A \leq \sup B$, *and* $\inf(A) \geq \inf(B)$.
2. $\sup(A \cup B) = \max\{\sup(A), \sup(B)\}$.
3. $\inf(A \cup B) = \min\{\inf(A), \inf(B)\}$.

Proof

1) $A \subset B \Rightarrow \sup(A) \leq \sup(B)$ and $\inf(A) \geq \inf(B)$

Let $x \in A \Rightarrow x \in B$ such that $x \leq \sup(B)$ hence $\sup(B)$ is upper bound of A therefore $\sup(A)$ is smallest upper bound of A then the desired result.
Is the same way we can prove $\inf(A) \geq \inf(B)$
Let $x \in A \Rightarrow x \in B$ such that $x \geq \inf(B)$ hence $\inf(B)$ is lower bound of A moreover $\inf(A)$ is the greatest lower bound therefore $\inf(A) \geq \inf(B)$

Proof

2) $\sup(A \cup B) = \max\{\sup(A), \sup(B)\}$. According to the previous question

$$A \subset A \cup B \Rightarrow \sup(A) \leq \sup(A \cup B)$$

$$B \subset A \cup B \Rightarrow \sup(B) \leq \sup(A \cup B)$$

Then $\max\{\sup(A), \sup(B)\} \leq \sup(A \cup B) \ldots.(*)$
Let $x \in A \cup B$ implies ($x \in A$ or $x \in B$) so ($x \leq \sup(A)$ or $x \leq \sup(B)$) hence $\sup(A)$, $\sup(B)$ are upper bounds for $A \cup B$ but $\sup(A \cup B)$ is smallest upper bound, so

$$\sup(A \cup B) \leq \sup(A) \quad \text{and, } \sup(A \cup B) \leq \sup(B)$$

then $\sup(A \cup B) \leq \max\{\sup(A), \sup(B)\} \ldots(**)$
From (*) and (**) we have the desired equality.

proof

3) $\inf(A \cup B) = \min\{\inf(A), \inf(B)\}$. According to the previous question

$$A \subset A \cup B \Rightarrow \inf(A) \geq \sup(A \cup B)$$

$$B \subset A \cup B \Rightarrow \inf(B) \geq \inf(A \cup B)$$

Then $\min\{\inf(A), \inf(B)\} \geq \inf(A \cup B)....(*)$
Let $x \in A \cup B$ implies ($x \in A$ or $x \in B$) so ($x \geq \inf(A)$ or $x \geq \inf(B)$) hence $\inf(A)$, $\inf(B)$ are lower bounds for $A \cup B$ but $\inf(A \cup B)$ is greatest lower bound, so

$$\inf(A \cup B) \geq \inf(A) \quad \text{and,} \ \inf(A \cup B) \geq \inf(B)$$

then $\inf(A \cup B) \geq \min\{\inf(A), \inf(B)\}...(**)$
From (*) and (**) we have the desired equality.

Intervals

We now introduce the following notation and terminology for the number sets listed below:

$$]a, b[= \{x \in \mathbb{R} : a < x < b\} \quad \text{is the open interval}$$

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\} \quad \text{is the closed interval}$$

$$]a, b] = \{x \in \mathbb{R} : a < x \leq b\} \quad \text{is the half open interval, containing b}$$

$$[a, b[= \{x \in \mathbb{R} : a \leq x < b\} \quad \text{is the half open interval, containing a}$$

Definition

Open, closed, and half-open intervals are called numerical intervals or simply intervals. The numbers determining an interval are called its endpoints.

Intervals

The quantity $b - a$ is called the length of the interval ab. If I is an interval, we shall denote its length by $|I|$.
The sets

$$]a, +\infty[= \{x \in \mathbb{R} : x > a\}, \quad [a, +\infty[= \{x \in \mathbb{R} : x \geq a\}$$

$$]\infty, b[= \{x \in \mathbb{R} : x < b\}, \quad]\infty, b] = \{x \in \mathbb{R} : x \leq b\},$$

and $] - \infty, +\infty[= \mathbb{R}$ are conventionally called unbounded intervals or infinite intervals.

Definition

An open interval containing the point $x \in \mathbb{R}$ will be called a neighborhood of this point.

Absolute value

So as not to have to investigate which of the points is "left" and which is "right", that is, whether $x < y$ or $y < x$ and whether the length is $y - x$ or $x - y$, we can use the useful function

$$|x| = \sup\{x, -x\}$$

or on other words

$$|x| = \begin{cases} x & \text{when } x > 0 \\ 0 & x = 0 \\ -x & \text{when } x < 0. \end{cases}$$

which is called the **modulus** or **absolute value** of the number.

Absolute value

Proposition

For all $x,\ y \in \mathbb{R}$ *we have:*

1. $|x| \geq 0, \quad |x| = -|x|.$
2. $|x| = 0$ *if and only if* $x = 0.$
3. $|x| = x$ *if and only if* $x \geq 0.$
4. $|xy| = |x||y|$
5. $|\dfrac{x}{y}| = \dfrac{|x|}{|y|}, \quad y \neq 0.$
6. *Let* $\alpha > 0$. *if* $|x| \leq \alpha$ *if and only if* $-\alpha \leq x \leq \alpha.$
7. $|x| \geq \alpha$ *if and only if* $x \in]-\infty, \alpha] \bigcup [\alpha, +\infty[.$
8. $\sqrt{x^2} = |x|, \quad |x|^2 = x^2$
9. $|x + y| \leq |x| + |y|$. *triangular inequality.*

Proof

- If $x \geq 0$ and $y \geq 0$, then $x + y \geq 0$, then $|x + y| = x + y$ and $|x| = x$ and $|y| = y$, so that equality holds in this case.
- If $x \leq 0$ and $y \leq 0$, then $x + y \leq 0$, then $|x + y| = -(x + y) = -x - y = -x + (-y)$ and $|x| = -x$ and $|y| = -y$, and again we have equality.
- Now suppose one of the numbers is negative and the other positive, for example, $x < 0 < y$. Then either $x < x + y \leq 0$ or $0 \leq x + y < y$.
  In the first case
  $|x + y| = -(x + y) = -x + (-y) = |x| - |y| \leq |x| + |y|$, and in the same from second case so that in both cases
  $$|x + y| \leq |x| + |y|.$$

Exercise

Prove that for all real numbers $x_1, x_2, ..., x_n$

$$|\sum_{i=1}^{n} x_i| \leq \sum_{i=1}^{n} |x_i| \text{ or } |x_1 + x_2 + ... + x_n| \leq |x_1| + |x_2| + ... + |x_n|$$

By Induction: $n = 1, 2$ the inequality is true
we assume that the inequality is true for all n and prove for $n + 1$

$$|\sum_{i=1}^{n+1} x_i| = |\sum_{i=1}^{n} x_i + x_{n+1}| \leq |\sum_{i=1}^{n} x_i| + |x_{n+1}| \leq \sum_{i=1}^{n} |x_i| + |x_{n+1}| = \sum_{i=1}^{n+1} |x_i|.$$

Exercise

Find if the exists the upper bounds, lower bounds supremun, infimum, maximum and minimum of the following set:

$$G = \{e^{-n}, \quad n \in \mathbb{N}\}, \quad C =]-1, 3[\cup]5.6].$$

We consider the the function $g(x) = e^{-x}, \quad x \in [0, +\infty[$.
g is decreasing over $[0, +\infty[$ into $]0, 1]$ so

$\sup(G) = 1, \quad \max(G) = 1, \quad \inf(G) = 0, \quad \min(G) =$ does not exist

- for $C = A \cup B$, such that $A =]-1, 3[, \quad B =]5.6]$

$$\sup(A \cup B) = \max(\sup(A), \sup(B)) = \max(3, 6) = 6.$$

$$\inf(A \cup B) = \min(\inf(A), \inf(B)) = \min(-1, 5) = -1.$$