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Sample Correction for Series N°04.
Algebra 1

Exercise 01 :

- Recall the definition of an internal composition law (binary operation) : An internal composition law on a set E is a function $* : E \times E \to E$ that assigns to each pair $(a, b) \in E \times E$ a unique element $a * b \in E$.

1. **Is $*$ commutative ? Associative ?**
   - **Commutativity :**
   $$x * y = x + y + x^2y^2$$
   $$y * x = y + x + y^2x^2 = x + y + x^2y^2$$
   Since $x * y = y * x$, the operation $*$ is commutative.
   - **Associativity :**
   $$(x * y) * z = (x + y + x^2y^2) * z$$
   $$= (x + y + x^2y^2) + z + (x + y + x^2y^2)^2z^2$$
   $$= x + y + z + x^2y^2 + (x + y + x^2y^2)^2z^2$$
   $$x * (y * z) = x * (y + z + y^2z^2)$$
   $$= x + (y + z + y^2z^2) + x^2(y + z + y^2z^2)^2$$
   $$= x + y + z + y^2z^2 + x^2(y + z + y^2z^2)^2$$
   Since $(x * y) * z \neq x * (y * z)$ in general, the operation $*$ is not associative.

2. **Determine the identity element of $*$.** - Let e be the identity element such that $x*e = x$ for all $x \in \mathbb{R}$.
   $$x * e = x + e + x^2e^2 = x$$
   $$e + x^2e^2 = 0$$
   - Solving for e :
   $$e(1 + x^2e) = 0$$
   - For all x, this implies $e = 0$. Therefore, the identity element of $*$ is 0.

- (a) 1- $\perp$ is associative $\Leftrightarrow \forall x, y, z \in \mathbb{R}, (x \perp y) \perp z = x \perp (y \perp z)$.
  $$\Leftrightarrow \forall x, y, z \in \mathbb{R}, a^2x + aby + bz = ax + aby + b^2z.$$
  Therefore, $a^2 = a$, $ab = ba$, and $b = b^2$. Hence, ($a = 0$ or $a = 1$) and ($b = 0$ or $b = 1$).

- 2- $\perp$ has a neutral element $e \in \mathbb{R}$ if $\forall x \in \mathbb{R}, x \perp e = e \perp x = x$.
  $$\Leftrightarrow \forall x \in \mathbb{R}, ax + be = ae + bx = x.$$
  $\Leftrightarrow a = 1$, $e = 0$, and $b = 1$.

Exercise 02 :

We define two internal composition laws on $\mathbb{R}$ (the set of real numbers) as follows :

$$x * y = x + y - 1$$

$$x \perp y = x + y - xy$$

1. **Show that $(\mathbb{R}, *)$ is a commutative group. - Commutativity :**

$$x * y = x + y - 1 = y + x - 1 = y * x$$

- **Associativity :**

$$(x * y) * z = (x + y - 1) + z - 1 = x + y + z - 2$$

$$x * (y * z) = x + (y + z - 1) - 1 = x + y + z - 2$$

Hence, $(x * y) * z = x * (y * z)$.
- **Identity element :**

$$x * e = x + e - 1 = x \Rightarrow e = 1$$

(By commutativity, we do not need to check $e * x$).
- **Inverse element :** For an element x, Let its inverse is $x^{'}$. Thus,

$$x * x^{'} = x + x^{'} - 1 = 1 \Rightarrow x^{'} = 2 - x$$

(By commutativity, we do not need to check $x^{'} * x$). Therefore, $(\mathbb{R}, *)$ is a commutative group.

2. **Prove that H is a subgroup of $(\mathbb{R}, +)$.**

i) Identity element of $+$ in H. The identity in $(\mathbb{R}, +)$ is 0. Choose $t = 0$. Then $0 = 2 \cdot 0 \in H$.

ii) H is Closure under $x + y'$. Let $x, y \in H$. Write $x = 2a$ and $y = 2b$ for some $a, b \in \mathbb{Z}$. The additive inverse of y is $y' = -y = -2b = 2(-b) \in H$. Hence

$$x + y' = 2a + (-2b) = 2(a - b) \in H.$$

Since $0 \in H$ and $x + y' \in H$ for all $x, y \in H$, it follows that H is a subgroup of $(\mathbb{R}, +)$.

3. **Show that the law $\perp$ is distributive with respect to the law $*$.**
- **Distributivity :**

$$x \perp (y * z) = x + (y + z - 1) - x(y + z - 1) = x + y + z - 1 - xy - xz + x$$

$$(x \perp y) * (x \perp z) = (x + y - xy) * (x + z - xz) = (x + y - xy) + (x + z - xz) - 1 = 2x + y + z - xy - xz - 1$$

- Since $x \perp (y * z) = (x \perp y) * (x \perp z)$ holds true, the law $\perp$ is distributive with respect to the law $*$.

4. **Show that** $(\mathbb{R}, *, \perp)$ **is a ring. Is it a field ?**
 - We have already shown that $(\mathbb{R}, *)$ is a commutative group and $\perp$ is distributive over $*$.
 - To be a ring, $\perp$ must also be associative. Since we defined it distributively, we have :

$$(x \perp y) \perp z = x + y - xy + z - xz - yz + xyz$$

$$x \perp (y \perp z) = x + y + z - xy - xz - yz + xyz$$

Therefore, $(\mathbb{R}, *, \perp)$ is associative and hence a ring.
 - To be a field, every non-zero element must have an inverse with respect to $\perp$. But due to our defined operations, not every element has an inverse. Therefore, $(\mathbb{R}, *, \perp)$ is a ring, but not a field.

Exercise 03 :

1. **Show that** $(\mathbb{R}, *)$ **is a commutative group, where** $x * y = \sqrt[3]{x^3 + y^3}$.

   - Commutativity : For all $x, y \in \mathbb{R}$,

   $$x * y = \sqrt[3]{x^3 + y^3} = \sqrt[3]{y^3 + x^3} = y * x.$$

   Therefore, $(\mathbb{R}, *)$ is a commutative group.
   - Associativity : For all $x, y, z \in \mathbb{R}$,

   $$(x*y)*z = \sqrt[3]{\left(\sqrt[3]{x^3 + y^3}\right)^3 + z^3} = \sqrt[3]{x^3 + y^3 + z^3} = \sqrt[3]{x^3 + \left(\sqrt[3]{y^3 + z^3}\right)^3} = x*(y*z).$$

   - Identity : Let $e \in \mathbb{R}$ satisfy $x * e = x$ for all x. Then

   $$\sqrt[3]{x^3 + e^3} = x \quad \Longrightarrow \quad e^3 = 0 \quad \Longrightarrow \quad e = 0.$$

   Thus the identity is 0, and $0 * x = x$ holds similarly.
   - Inverse Property : Let $x \in \mathbb{R}$, we seek $x' \in \mathbb{R}$ with $x * x' = 0$. This is equivalent to

   $$\sqrt[3]{x^3 + (x')^3} = 0 \quad \Longleftrightarrow \quad x^3 + (x')^3 = 0 \quad \Longleftrightarrow \quad x' = -x.$$

   Hence the inverse of x in $(\mathbb{R}, *)$ is $-x$.

2. Let $f : (\mathbb{R}, +) \to (\mathbb{R}, *)$ be defined by $f(x) = \sqrt[3]{x}$.

   (a) $\mathrm{Ker}(f) = \{x \in \mathbb{R}, f(x) = 0\} = \{x \in \mathbb{R}, \sqrt[3]{x} = 0\} = \{0\}$, $\quad \mathrm{Im}(f) = \{f(x), x \in \mathbb{R}\} = \mathbb{R}$.

   (b) Homomorphism :

   $$f(x + y) = \sqrt[3]{x + y} = \sqrt[3]{\left(\sqrt[3]{x}\right)^3 + \left(\sqrt[3]{y}\right)^3} = f(x) * f(y).$$

   (c) Isomorphism : Since $\mathrm{Ker}(f) = \{0\}$ then f is injective ; since $\mathrm{Im}(f) = \mathbb{R}$ then f is surjective. Therefore f is bijective and, being a homomorphism, it is an isomorphism from $(\mathbb{R}, +)$ onto $(\mathbb{R}, *)$.