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**Sample Correction of Exercise Series N°03.**
**Algebra 1**

**Correction of Exercise 01 :**

1. False. The negation of the definition of a symmetric relation just differs from the one of an anti-symmetric relation .
2. False. A binary relation may be symmetric and anti-symmetric simultaneously. For instance, one may consider a relation $R$ defined on $\mathbb{R}$, by $xRy \Leftrightarrow x = y$. Alternatively, you may consider any relation on a singleton.
3. True. Indeed, if $xRy$, then given the fact that the relation is symmetric, we get $yRx$ too. Since $R$ is also anti-symmetric, we obtain $x = y$.
4. True. This is just a reformulation of the definition of an anti-symmetric relation using the contrapositive.
5. False. For example, define a binary relation $R$ on $\mathbb{R}$ by : $xRy \Leftrightarrow xy \neq 0$. Then $R$ is both symmetric and transitive, yet it is not reflexive since for $x = 0$, it is seen that $x \not{R} x$.

**Correction of Exercise 02 :**

1. (a) $R$ is not reflexive since $3 \not{R} 3$.
   (b) $R$ is not symmetric since $4R2$ and $2 \not{R} 4$.
   (c) $R$ is not anti-symmetric as $2R3$ and $3R2$ yet $2 \neq 3$.
   (d) $R$ is not transitive because $4R2$ and $2R3$ yet $4 \not{R} 3$.
2. (a) $R$ is not reflexive : If it were, we would have $\forall x \in \mathbb{Z} : xRx$, i.e. $\forall x \in \mathbb{Z} : x = -x$. But $\exists x = 1 \in \mathbb{Z}$ such that $1 = x \neq -x = -1$. Hence, $R$ is not reflexive.
   (b) $R$ is symmetric because for all $x, y \in \mathbb{Z} : xRy \Leftrightarrow x = -y \Rightarrow y = -x \Leftrightarrow yRx$.
   (c) $R$ is not anti-symmetric because $\exists 1, -1 \in \mathbb{Z} : 1R(-1)$ and $(-1)R1$ yet $1 \neq -1$.
   (d) $R$ is not transitive : For example, $\exists 1, -1 \in \mathbb{Z} : 1R(-1)$ and $(-1)R1$ but $1 \not{R} 1$.
3. (a) $R$ is reflexive because $\forall x \in \mathbb{R} : e^x \leq e^x$.
   (b) $R$ is not symmetric because $\exists 0, 1 \in \mathbb{R} : e^0 = 1 \leq e^1$ yet $e^1 \nleq e^0$.
   (c) $R$ is anti-symmetric because $\forall x, y \in \mathbb{R} : xRy \wedge yRx \Leftrightarrow e^x \leq e^y \wedge e^y \leq e^x \Rightarrow e^x = e^y \Leftrightarrow x = y$.
   (d) $R$ is transitive because $\forall x, y, z \in \mathbb{R} : xRy \wedge yRz \Leftrightarrow e^x \leq e^y \wedge e^y \leq e^z \Rightarrow e^x \leq e^z \Leftrightarrow xRz$.
4. (a) The relation $R$ is reflexive. Let $x \in \mathbb{R}$. The following trigonometric formula is well-known : $\cos^2 x + \sin^2 x = 1$. It is equivalent to $xRx$, i.e. $R$ is reflexive.

(b) $R$ is symmetric : To see this, let $x, y \in \mathbb{R}$ be such that $xRy$, i.e. $\cos^2 y + \sin^2 x = 1$. Since $\forall x, y \in \mathbb{R} : \cos^2 y + \sin^2 y = 1$ and $\cos^2 x + \sin^2 x = 1$, it follows that $\cos^2 y + \sin^2 x = 1 \Leftrightarrow \cos^2 y + \sin^2 x = \cos^2 y + \sin^2 y$ or $\sin^2 y = \sin^2 x$. Likewise $\cos^2 y + \sin^2 x = 1 \Leftrightarrow \cos^2 y + \sin^2 x = \cos^2 x + \sin^2 x$ yields $\cos^2 y = \cos^2 x$. Finally, we get $\cos^2 y + \sin^2 x = 1 \Leftrightarrow \cos^2 x + \sin^2 y = 1 \Leftrightarrow yRx$, i.e. $R$ is symmetric.

(c) $R$ is not anti-symmetric : Recall that this signifies that $\exists x, y \in \mathbb{R} : xRy \wedge yRx$ but $x \neq y$. For instance, $0R(2\pi)$ because $\cos^2(2\pi) + \sin^2 0 = 1 + 0 = 1$, and $(2\pi)R0$ for $\cos^2 0 + \sin^2(2\pi) = 1 + 0 = 1$, and yet $0 \neq 2\pi$.

(d) $R$ is transitive : Let $x, y, z \in \mathbb{R}$ and suppose that $xRy$ and $yRz$, i.e. $\cos^2 y + \sin^2 x = 1$ and $\cos^2 z + \sin^2 y = 1$. Hence $\cos^2 y + \sin^2 x + \cos^2 z + \sin^2 y = 1 + 1 = 2$. But $\sin^2 x + \cos^2 z + \cos^2 y + \sin^2 y = 1 = 2$, and so $\cos^2 z + \sin^2 x = 1$, i.e. $xRz$, i.e. $R$ is transitive.

5. We conclude that relation 3 is an order relation and relation 4 is an equivalence relation.

**Correction of Exercise 03 :**

1. That $R$ is an equivalence relation is left to the students.

2. In order that $R$ be an order relation, it has to be anti-symmetric, given that it is already reflexive and transitive (by the previous question). Recall that $R$ is anti-symmetric if for all $x, y \in R$, $xRy$ and $yRx$, we obtain $x = y$. Put differently, for all $x, y \in R$, we must have $f(x) = f(y) \Rightarrow x = y$. It then becomes clear that the injectivity of $f$ resolves this issue.

3. A relation $R$ on a set $X$ is not total order. Let take the following counter example.$f : \mathbb{R} \longrightarrow \mathbb{R}$; $f(x) = x$. We obtain that $f$ is injective but for any $x, y \in X$, $f(x) \neq f(y)$ neither $xRy$ nor $yRx$ holds.

**Correction of Exercise 04 :**

1. (a) Let $x \in \mathbb{Z}$. Since $x + x = 2x$ is always even, $R$ is reflexive.

   (b) Let $x, y \in \mathbb{Z}$. The obvious fact that $x + y$ is even if and only if $y + x$ is so, amply means that $R$ is symmetric.

   (c) The relation $R$ is transitive. To prove this, let $x, y, z \in \mathbb{Z}$ and assume that $xRy$ and $yRz$, i.e. $x + y$ and $y + z$ are even. So, there exist $n, m \in \mathbb{Z}$ such that $x + y = 2n$ and $y + z = 2m$. Thus,

$$x + y + y + z = 2n + 2m \Rightarrow x + z = 2(n + m - y) \in \mathbb{Z},$$

   i.e. $x + z$ is even, that is, $xRz$

2. Observe that with respect to $R$ here, any even number is in relation with another even number, as is any odd number with another odd number. More precisely, we have :

$$[0] = \{y \in \mathbb{Z} : 0Rz\} = \{y \in \mathbb{Z} : 0 + y \text{ even}\} = \{0, \pm 2, \pm 4, \ldots\}$$

and

$$[1] = \{y \in \mathbb{Z} : 1Rz\} = \{y \in \mathbb{Z} : 1 + y \text{ even}\} = \{\pm 1, \pm 3, \pm 5, \ldots\}.$$

These two equivalence classes, { [0], [1] }, cover all the integers without any overlap. Since $[0] \cup [1] = \mathbb{Z}$ and $[0] \cap [1] = \emptyset$, these classes partition $\mathbb{Z}$ completely. Hence, the partition set of the total set $\mathbb{Z}$ is { [0], [1] }.