# Sample Correction for Series N°01.
# Algebra 1

**Exercise 01 :**

1. **Not a proposition**. This is an open sentence. It becomes a proposition when $x$ is quantified : $\forall x \in \mathbb{R}, x^2 \geq 0$ is true, $\exists x \in \mathbb{R}, x^2 \geq 0$ is true.
2. **True proposition**. Mathematical proof : $\forall x \in \mathbb{R}, x^2 \geq 0 \Rightarrow x^2 + 1 \geq 1 > 0$, so the equation $x^2 + 1 = 0$ has no real solution.
3. **True proposition**. In $\mathbb{R}$, $x^2 \geq 0$ for all $x$, so $x^2 = -1$ is impossible. The solution exists only in $\mathbb{C}$ : $x = \pm i$.
4. **True proposition**. If $x^2 = 9$ then $x = 3$ or $x = -3$. Mathematical proof : $x^2 = 9 \Leftrightarrow x^2 - 9 = 0 \Leftrightarrow (x-3)(x+3) = 0$.
5. **True proposition**. The number 2 is prime and even. Proof : 2 is divisible only by 1 and itself, and 2 is even.
6. **False proposition**. The interval $[0, 1]$ is uncountably infinite.
7. **True proposition**. This is the definition of a triangle in Euclidean geometry : a polygon with 3 vertices and 3 sides.
8. **True proposition**. This is the Pythagorean theorem : $c^2 = a^2 + b^2$ where $c$ is the hypotenuse.
9. **False proposition**. Counterexamples :
   - Rectangle : 4 right angles
   - Square : 4 right angles
   - Rhombus : 0 right angles
10. **False proposition**. Counterexample : A rectangle with sides 2 and 3 is not a square. A square requires all sides equal.
11. **True proposition**. Every square satisfies the rectangle definition : quadrilateral with four right angles.
12. **True proposition**. $\pi$ is irrational (proven by Lambert in 1761 using continued fractions).
13. **False proposition**. The circumference is $C = 2\pi r$, not $2\pi$. Missing the radius factor. However, the angular measure of any circle's circumference is $2\pi$ radians.
14. **Not a proposition**. This is ambiguous and context-dependent. "I'm reading" does not logically imply "I am in the classroom" - one can read anywhere.

15. **Not a proposition**. Vague and unverifiable without specifying what "he is saying". Also creates a paradox if "he" says "this statement is false".

16. **Not a proposition**. This is an imperative sentence (command), not a declarative statement with truth value.

17. **True proposition**. All students must pass the baccalaureate exam to graduate to university level.

18. **Not a proposition**. This is an interrogative sentence (question), not a statement that can be true/false.

19. **Not a proposition**. This is a subjective aesthetic judgment.

20. **Not a proposition**. This is an expressive sentence (wish/desire).

**Exercise 02 :**
**Solution for Part i**

1. Truth table for $\bar{p} \land \bar{q}$ :

| $p$ | $q$ | $\bar{p}$ | $\bar{q}$ | $\bar{p} \land \bar{q}$ |
|---|---|---|---|---|
| T | T | F | F | F |
| T | F | F | T | F |
| F | T | T | F | F |
| F | F | T | T | T |

2. Truth table for $\bar{p} \lor \bar{q}$ :

| $p$ | $q$ | $\bar{p}$ | $\bar{q}$ | $\bar{p} \lor \bar{q}$ |
|---|---|---|---|---|
| T | T | F | F | F |
| T | F | F | T | T |
| F | T | T | F | T |
| F | F | T | T | T |

3. Truth table for $\overline{p \land q}$ :

| $p$ | $q$ | $p \land q$ | $\overline{p \land q}$ |
|---|---|---|---|
| T | T | T | F |
| T | F | F | T |
| F | T | F | T |
| F | F | F | T |

4. Truth table for $\overline{p \lor q}$ :

| $p$ | $q$ | $p \lor q$ | $\overline{p \lor q}$ |
|---|---|---|---|
| T | T | T | F |
| T | F | T | F |
| F | T | T | F |
| F | F | F | T |

**Solution for Part ii**
From the truth tables :

- Comparing (1) and (4) : $\bar{p} \wedge \bar{q} \equiv \overline{p \vee q}$ (De Morgan's Law)
- Comparing (2) and (3) : $\bar{p} \vee \bar{q} \equiv \overline{p \wedge q}$ (De Morgan's Law)

**Conclusion :** We have verified De Morgan's Laws :

$$\overline{p \wedge q} \equiv \bar{p} \vee \bar{q} \quad \text{and} \quad \overline{p \vee q} \equiv \bar{p} \wedge \bar{q}$$

**Solution for Part iii**

1. Truth table for $(p \wedge (q \vee r)) \Leftrightarrow ((p \wedge q) \vee (p \wedge r))$ :

| $p$ | $q$ | $r$ | $q \vee r$ | $p \wedge (q \vee r)$ | $p \wedge q$ | $p \wedge r$ | $(p \wedge q) \vee (p \wedge r)$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T |
| T | T | F | T | T | T | F | T |
| T | F | T | T | T | F | T | T |
| T | F | F | F | F | F | F | F |
| F | T | T | T | F | F | F | F |
| F | T | F | T | F | F | F | F |
| F | F | T | T | F | F | F | F |
| F | F | F | F | F | F | F | F |

The equivalence column would show all T, proving the distributive law.

2. Truth table for $((p \vee q) \vee r) \Leftrightarrow (p \vee (q \vee r))$ :

| $p$ | $q$ | $r$ | $p \vee q$ | $(p \vee q) \vee r$ | $q \vee r$ | $p \vee (q \vee r)$ |
|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T |
| T | T | F | T | T | T | T |
| T | F | T | T | T | T | T |
| T | F | F | T | T | F | T |
| F | T | T | T | T | T | T |
| F | T | F | T | T | T | T |
| F | F | T | F | T | T | T |
| F | F | F | F | F | F | F |

Both sides are identical, proving the associative law for disjunction.

3. Truth table for $(p \Rightarrow q) \Leftrightarrow (\bar{p} \vee q)$ :

| $p$ | $q$ | $p \Rightarrow q$ | $\bar{p}$ | $\bar{p} \vee q$ | Equivalence |
|---|---|---|---|---|---|
| T | T | T | F | T | T |
| T | F | F | F | F | T |
| F | T | T | T | T | T |
| F | F | T | T | T | T |

All entries in the equivalence column are T, proving the equivalence.

**Exercise 03 :**

a) **Original Form :** For all real number $x$, the function $f$ is positive.
**Quantifiers Form :** $\forall x \in \mathbb{R},\ f(x) > 0$
**Negation Form :** $\exists x \in \mathbb{R},\ f(x) \leq 0$

b) **Original Form :** For all real number $x$, the function $f$ is increasing.
**Quantifiers Form :** $\forall x, y \in \mathbb{R},\ (x < y \Rightarrow f(x) < f(y))$
**Negation Form :** $\exists x, y \in \mathbb{R},\ (x < y \wedge f(x) \geq f(y))$ (use the negation rule of the implication)

c) **Original Form :** For all real number $x$, the function $f$ is constant.
**Quantifiers Form :** $\forall x, y \in \mathbb{R},\ f(x) = f(y)$
**Negation Form :** $\exists x, y \in \mathbb{R},\ f(x) \neq f(y)$

d) **Original Form :** The sum of two even integer numbers $x$ and $y$ is even.
**Quantifiers Form :** $\forall x, y \in \mathbb{Z},\ (x = 2k_1 \wedge y = 2k_2) \Rightarrow x+y = 2k,$ with $k_1, k_2$ and $k \in \mathbb{Z}$
**Negation Form :** $\exists x, y \in \mathbb{Z},\ (x = 2k_1 \wedge y = 2k_2) \wedge x + y = 2k + 1,$ with $k_1, k_2$ and $k \in \mathbb{Z}$ (False Proposition)

e) **Original Form :** The function $f$ is zero at some real numbers.
**Quantifiers Form :** $\exists x \in \mathbb{R},\ f(x) = 0$
**Negation Form :** $\forall x \in \mathbb{R},\ f(x) \neq 0$

f) **Original Form :** The function $f$ vanishes at a unique real number $a$.
**Quantifiers Form :** $\exists! a \in \mathbb{R},\ f(a) = 0$
**Negation Form :** $[\forall a \in \mathbb{R},\ f(a) \neq 0] \vee [\exists x \in \mathbb{R}(f(x) = 0 \wedge x \neq a)]$

**Exercise 04 :**

1. $\forall \alpha \in \mathbb{R}, \forall \beta \in \mathbb{R}, \alpha + \beta > 0$. **False**
   **Counterexample :** Take $\alpha = -2$ and $\beta = -1$, then $\alpha + \beta = -3 \leq 0$
   **Negation :** $\exists \alpha \in \mathbb{R}, \exists \beta \in \mathbb{R}, \alpha + \beta \leq 0$

2. $\exists \alpha \in \mathbb{R}, \forall \beta \in \mathbb{R}, \alpha + \beta > 0$. **False**
   **Proof :** Suppose such $\alpha$ exists. Then for $\beta = -\alpha - 1$, we have :
   $$\alpha + \beta = \alpha + (-\alpha - 1) = -1 \leq 0$$
   Contradiction. Therefore, no such $\alpha$ exists.
   **Negation :** $\forall \alpha \in \mathbb{R}, \exists \beta \in \mathbb{R}, \alpha + \beta \leq 0$

3. $\forall \alpha \in \mathbb{R}, \forall \beta \in \mathbb{R}, \alpha^2 + \beta^2 \geq 0$. **True**
   **Proof :** For all $\alpha, \beta \in \mathbb{R}$, we have $\alpha^2 \geq 0$ and $\beta^2 \geq 0$. Therefore :
   $$\alpha^2 + \beta^2 \geq 0$$
   Equality holds only when $\alpha = \beta = 0$.
   **Negation :** $\exists \alpha \in \mathbb{R}, \exists \beta \in \mathbb{R}, \alpha^2 + \beta^2 < 0$

4. $\forall \alpha \in \mathbb{R}, |\alpha| \geq 0$. **True**

   **Proof :** By definition of absolute value :

$$|\alpha| = \begin{cases} \alpha & \text{if } \alpha \geq 0 \\ -\alpha & \text{if } \alpha < 0 \end{cases}$$

   In both cases, $|\alpha| \geq 0$. Equality holds when $\alpha = 0$.

   **Negation :** $\exists \alpha \in \mathbb{R}, |\alpha| < 0$

5. $\exists \alpha \in \mathbb{R}, \forall \beta \in \mathbb{R}, \beta^2 > \alpha$ $(*)$. **True**

   **Proof :** Take $\alpha = -1$. Then for all $\beta \in \mathbb{R}$, we have $\beta^2 \geq 0 > -1 = \alpha$.

   **Negation :** $\forall \alpha \in \mathbb{R}, \exists \beta \in \mathbb{R}, \beta^2 \leq \alpha$

**Exercise 05 :**

1. **Proof (Direct Verification) :**. $\exists x \in [-1, 1],\ x^2 - \frac{1}{4} = 0$. **True**

$$x^2 - \frac{1}{4} = 0 \Rightarrow x^2 = \frac{1}{4} \Rightarrow x = \pm\frac{1}{2}$$

   Both $x = \frac{1}{2}$ and $x = -\frac{1}{2}$ are in $[-1, 1]$.

2. **Direct proof :**. The product of two odd integer numbers $x$ and $y$ is odd. **True**

   Let $x = 2k + 1$, $y = 2m + 1$ where $k, m \in \mathbb{Z}$. Then :

$$x \cdot y = (2k + 1)(2m + 1) = 4km + 2k + 2m + 1 = 2(2km + k + m) + 1$$

   This is of the form $2p + 1$ where $p \in \mathbb{Z}$, hence odd.

3. **Proof by counterexample.** All prime numbers are odd. **False**

   The number 2 is prime and even.

4. **Proof by contradiction :** $\forall a, b \in \mathbb{R}^+$, if $\frac{b+1}{a} = \frac{a+1}{b} \Rightarrow a = b$. **True**

   - Assume the implication is false. That is its negation is true, i.e,

$$\frac{b+1}{a} = \frac{a+1}{b} \quad \text{and} \quad a \neq b, \quad \forall a, b \in \mathbb{R}^+.$$

   - From $\frac{b+1}{a} = \frac{a+1}{b}$, cross-multiplying gives :

$$b(b + 1) = a(a + 1)$$

$$\Leftrightarrow b^2 + b = a^2 + a$$

$$\Leftrightarrow b^2 - a^2 + b - a = 0$$

$$\Leftrightarrow (b - a)(b + a) + (b - a) = 0$$

$$\Leftrightarrow (b - a)(b + a + 1) = 0$$

- Since $a, b > 0$, we have $b + a + 1 > 0$. Therefore, we must have $b - a = 0 \Rightarrow a = b$. But this **contradicts** our assumption that $a \neq b$.
  Therefore, the original implication must be true

5. **Proof by Induction :** $\forall n \in \mathbb{N}^*,\ 1 + 3 + 5 + \ldots + (2n - 1) = \sum_{i=1}^{n}(2i - 1) = n^2$. **True**

   - Prove that true for initial case $(n = 1)$ :

   $$\sum_{i=1}^{1}(2i - 1) = 1 = 1^2 \quad \checkmark$$

   - Assume that true for $n$ :

   $$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

   - Prove that true for $(n + 1)$ :

   $$\begin{aligned} 1 + 3 + 5 + \cdots + (2n - 1) + (2(n + 1) - 1) &= n^2 + (2n + 1) \\ &= n^2 + 2n + 1 = (n + 1)^2 \end{aligned}$$

   - **Conclusion :** By mathematical induction, the formula holds for all $n \in \mathbb{N}^*$.