Exercise Series N°04.
Algebra 1

Exercise 01 :

- Recall the definition of an **internal composition law** (binary operation).

Let $*$ be a binary operation on $\mathbb{R}$ defined by :

$$\forall x, y \in \mathbb{R}, \quad x * y = x + y + x^2y^2.$$

1. Is $*$ commutative ? Associative ?
2. Determine the identity element of $*$.

Let $a, b \in \mathbb{R}$. We define an internal composition law $\perp$ on $\mathbb{R}$ as follows :

$$\forall x, y \in \mathbb{R}, \quad x \perp y = ax + by.$$

(a) Determine a and b so that the law $\perp$:

   (1) Is associative.

   (2) Has an identity element.

Exercise 02 :

We define two internal composition laws on $\mathbb{R}$ (the set of real numbers) as follows :

$$x * y = x + y - 1$$

$$x \perp y = x + y - xy$$

1. Show that $(\mathbb{R}, *)$ is **a commutative group**.
2. Prove that
$$H = \{\, x \in \mathbb{R} \mid x = 2t,\ t \in \mathbb{Z} \,\}$$
is a subgroup of $(\mathbb{R}, +)$ with $+$ is the usual addition operation.
3. Show that the law $\perp$ is **distributive** with respect to the law $*$.
4. Show that $(\mathbb{R}, *, \perp)$ is **a ring**. Is it **a field** ?

Exercise 03 : (Note : The solution will be discussed in the lecture session.)
Let $*$ be a binary operation on $\mathbb{R}$ defined by :

$$\forall x, y \in \mathbb{R}, \quad x * y = (x^3 + y^3)^{1/3}.$$

1. Show that $(\mathbb{R}, *)$ is a commutative group.
2. Show that the function f defined by

$$f : (\mathbb{R}, +) \to (\mathbb{R}, *)$$
$$x \mapsto f(x) = x^{1/3}$$

   (a) Determine $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$.
   (b) Prove that f is a homomorphism.
   (c) Is f an isomorphism ?

Exercise 04 : (Home Work)
Let $*$ be a binary operation on $E = \mathbb{R} \times \mathbb{R}^*$ defined by :

$$\forall a, c \in \mathbb{R},\ b, d \in \mathbb{R}^*, \quad (a, b) * (c, d) = (a + c + 1, bd).$$

1. Show that $(E, *)$ is a commutative group.
2. Let $m \in \mathbb{N}^*$. Let f_m be the function defined by

$$f_m : \mathbb{R} \times \mathbb{R}^* \to \mathbb{R}^*$$
$$(x, y) \mapsto f_m(x, y) = y^m.$$

   (a) Is f_m injective ? For which values of m is f_m surjective ?
3. Calculate $f_m([0, 1] \times [0, 2])$ and $f_m^{-1}(\{0\})$.
4. Show that f_m is a homomorphism from the group $(E, *)$ to the group $(\mathbb{R}^*, \times)$.
5. Let g be the homomorphism defined by

$$g : (\mathbb{R}^*, \times) \to (\mathbb{R}^*, \times)$$
$$x \mapsto x^2.$$

   Determine $g \circ f$. Show that $g \circ f$ is a homomorphism from $(E, *)$ to $(\mathbb{R}^*, \times)$.