Exercise Series N°03.
Algebra 1

Exercise 01 :
Determine, giving reasons, whether the following statements are true or false.

1. If a binary relation R is not symmetric, then it is anti-symmetric.
2. Any binary relation R is either symmetric or anti-symmetric.
3. Let R be a binary relation on some non-empty set E. If R is both symmetric and anti-symmetric, then $x \neq y$ implies that $x \not R\, y$ (where $x, y \in E$).
4. Let E be a non-empty set. If R is an anti-symmetric binary relation on E, and for any $x, y \in E$, $x\ R\ y$ and $x \neq y$, then $y \not R\, x$.
5. Let E be a set and R be a binary relation supposed symmetric and transitive. Let $x, y \in E$ be such that $x\ R\ y$. By symmetry, we have $y\ R\ x$ and by transitivity we get $x\ R\ x$, i.e., R is reflexive. Hence, a symmetric and transitive relation is necessarily reflexive.

Exercise 02 :
Decide whether each of the following binary relations are reflexive, symmetric, anti-symmetric, or transitive :

1. On the set $A = \{1, 2, 3, 4\}$, we define a binary relation R as follows :
$$1R1, 2R2, 2R3, 3R2, 4R2, 4R4.$$
2. On $\mathbb{Z}$, define :
$$xRy \iff x = -y.$$
3. On $\mathbb{R}$, define :
$$xRy \iff \exp(x) \leq \exp(y).$$
4. On $\mathbb{R}$, define :
$$xRy \iff \cos^2(y) + \sin^2(x) = 1.$$
5. What can we conclude about relation 3 and 4 ?

Exercise 03 :
Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We define on $\mathbb{R}$ a binary relation as follows : $xRy \iff f(x) = f(y)$.

1. Show that R is an equivalence relation.
2. Give a condition on f so that R becomes an order relation.
3. Is R a total order ?

Exercise 04 :
On $\mathbb{Z}$, we define an equivalence relation R by : $xRy \iff x+y$ is even.

1. Show that R is an equivalence relation.
2. What are the equivalence classes of this relation, i.e., $\mathbb{Z}_{/R}$? Deduce the partition set of the total set $\mathbb{Z}$.

Exercise 05 : (Home Work)
On $\mathbb{Z}$, we define a binary relation R by : $xRy \iff x-y$ is divisible by 4.

1. Show that R is an equivalence relation.
2. Find all equivalence classes of this relation.

Remark : Recall that $x-y$ is divisible by p ($p \in \mathbb{N}$) may be stated as : x is congruent to y modulo p. In symbols, $x \equiv y \mod p$.

Exercise 06 : (Home Work)
Over $\mathbb{R}^2$, define a binary relation, noted $\preceq$, by :

$$(x,y) \preceq (x',y') \iff x \leq x' \text{ and } y \leq y'.$$

1. Show that $\preceq$ is an order relation.
2. Is this a total order?