Exercise Series N°02.
Algebra 1

Note : Questions marked with $(*)$ are left for students.

Exercise 01 :
Among the following sets, determine which are well-defined sets, empty sets, or not well-defined sets. For the well-defined sets, determine their elements when possible.

1. $A = \{x \in \mathbb{R}, x^2 \geq 0\}$
2. $B = \{x \in \mathbb{R}, x^2 \leq 1\}$
3. $C = \{x \in \mathbb{R}, x^2 = 1\}$
4. $D = \{x \in \mathbb{R}, x^2 + 1 = 0\}$
5. $E = \{\text{beautiful flowers in the garden}\}$
6. $F = \{x \in \mathbb{Z}, -2 \leq x < 4\}$
7. $G = \{x \in \mathbb{R}, x < x\}$
8. $H = \{\text{the best students in the class}\}$
9. $I = \{x \in \mathbb{R}, x^3 - x = 0\}$
10. $J = \{x \in \mathbb{C}, x^2 = -4\}$. $(*)$
11. $K = \{p, p \text{ is a polygon with 2 sides}\}$
12. L is the set of large numbers.
13. $M = \{n \in \mathbb{N}, n \text{ is even and prime}\}$
14. $N = \{(x, y) \in \mathbb{R}^2, x^2 + y^2 = 1\}$. $(*)$
15. $O = \{y \in \mathbb{R}, |y| \leq 3\}$
16. $P = \{x \in \mathbb{R}, \sin x = 2\}$

Exercise 02 :

1. Let $E = \{-2, -1, 0, 1, 2, 3, a, e, f\}$. Are the following statements true? Justify your answers.
   - $0 = \emptyset$, $1 \subset E$, $\emptyset \in E$, $\emptyset \subset E$, $\emptyset \cup E = E$, $\{\} \cap E = \emptyset$.
2. Let $B = \{-1, 0, e, f\}$, $C = \{x \in \mathbb{R}, -2 \leq x < 3\}$ and $D = \{a, c, e\}$.
   (a) Determine : $B \cap D$, $B \cup D$, $C_E(B)$, $C_E(D)$, $E - B$, and $B \triangle D$.
   (b) $B \cap C$, $B \cup C$, $C_E(C)$, and $B \triangle C$. $(*)$
   (c) Determine : $B \times D$, $B \times \emptyset$, $\mathcal{P}(B)$, and $\mathcal{P}(D)$.
   (d) Can we write $C_D(B)$? Justify your answer.

Exercise 03 :
Let A, B, C be three subsets of the set E. Show that :

1. $A \cup B = A \cap B \Rightarrow A = B$
2. $A \cap B = \emptyset \Leftrightarrow A \subset C_E(B)$
3. $A \subset B \Leftrightarrow C_E(B) \subset C_E(A)$
4. $C_E(C_E(A)) = A$
5. $C_E(A \cup B) = C_E(A) \cap C_E(B)$
6. $C_E(A \cap B) = C_E(A) \cup C_E(B)$ (*)
7. $A \triangle B = C_E(A) \triangle C_E(B)$
8. $(A \times C) \cup (B \times C) = (A \cup B) \times C$
9. $A \subset B \Rightarrow \mathcal{P}(A) \subset \mathcal{P}(B)$ (*)

Exercise 04 :

Let E and F be two sets, and $f : E \to F$ be a function. Show that :

1. $\forall A, B \in \mathcal{P}(E),\ A \subset B \Rightarrow f(A) \subset f(B)$
2. $\forall A, B \in \mathcal{P}(E),\ f(A \cap B) \subset f(A) \cap f(B)$
3. $\forall A, B \in \mathcal{P}(E),\ f(A \cup B) = f(A) \cup f(B)$ (*)
4. $\forall\, C, D \in \mathcal{P}(F),\ f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$
5. $\forall\, D \in \mathcal{P}(F),\ f^{-1}(F \setminus D) = E \setminus f^{-1}(D)$ (*)

Exercise 05 :

Let :

$$\begin{array}{rcl} f : \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x) = |x| \end{array}$$

1. Is f injective ? Surjective ? (Justify).
2. Let $A = [-2, 0]$ and $B = [0, 1]$. Calculate : $f(A)$ and $f^{-1}(B)$.

Exercise 06 :

1. Let $E = [0, 1]$, $F = [-1, 1]$, and $G = [0, 2]$ be three intervals in $\mathbb{R}$. Consider the function f from E to G defined by :

   $$\begin{array}{rcl} f : E & \longrightarrow & G \\ x & \longmapsto & f(x) = 2 - x \end{array}$$

   and the function g from F to G defined by :

   $$\begin{array}{rcl} g : F & \longrightarrow & G \\ x & \longmapsto & g(x) = x^2 + 1 \end{array}$$

   (a) Determine $f(1/2)$, $f^{-1}(\{0\})$, $g([-1, 1])$, and $g^{-1}([0, 2])$.
   (b) Is the function f bijective ? Justify.
   (c) Is the function g bijective ? Justify.

2. Let h and k be the two applications from $\mathbb{R}$ to $\mathbb{R}$ defined by :

   $$h(x) = x^2 - 1, \quad k(x) = \cos(x)$$

   (a) Calculate : $(h \circ k)(x)$ and $(k \circ h)(x)$.
   (b) What do you conclude about $(h \circ k)(x)$ and $(k \circ h)(x)$?

3. Let $f(x) = \dfrac{x}{x+1}$. Determine :

   $$(f \circ f \circ \cdots \circ f)\,(x) \quad \text{based on } n \in \mathbb{N}^* \text{ and } x \in \mathbb{R}.$$