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Sample Correction of Exam .
Algebra 1

Exercise 1 (5 pts)

1. **Truth Table** We prove $(\overline{p} \vee q) \wedge \overline{r} \Leftrightarrow \overline{(p \wedge \overline{q}) \vee r}$ using a truth table. (**2 pt**)

p	q	r	$\overline{p}$	$\overline{q}$	$\overline{r}$	$\overline{p} \vee q$	$(\overline{p} \vee q) \wedge \overline{r}$	$p \wedge \overline{q}$	$(p \wedge \overline{q}) \vee r$	$\overline{(p \wedge \overline{q}) \vee r}$	Equivalence
T	T	T	F	F	F	T	**F**	F	T	**F**	**T**
T	T	F	F	F	T	T	**T**	F	F	**T**	**T**
T	F	T	F	T	F	F	**F**	T	T	**F**	**T**
T	F	F	F	T	T	F	**F**	T	T	**F**	**T**
F	T	T	T	F	F	T	**F**	F	T	**F**	**T**
F	T	F	T	F	T	T	**T**	F	F	**T**	**T**
F	F	T	T	T	F	T	**F**	F	T	**F**	**T**
F	F	F	T	T	T	T	**T**	F	F	**T**	**T**

2. **Quantified Form and Negation**
   Quantified form : (0.5 pt)

$$\forall x \in \mathbb{Z},\ (x-1)^2 = 2k \text{ with } k \in \mathbb{Z} \quad \Rightarrow \quad x = 2m+1 \text{ with } m \in \mathbb{Z}.$$

   Negation : (0.5 pt)

$$\exists x \in \mathbb{Z},\ (x-1)^2 = 2k \text{ with } k \in \mathbb{Z} \quad \wedge \quad x = 2m \text{ with } m \in \mathbb{Z}.$$

3. **Contrapositive Law : (1 pt)** For propositions p, q :

$$(p \Rightarrow q) \Leftrightarrow (\overline{q} \Rightarrow \overline{p}).$$

4. **Proof via Contrapositive (1 pt)**
   Contrapositive form :

$$x = 2m \text{ with } m \in \mathbb{Z} \ \Rightarrow\ (x-1)^2 = 2k+1 \text{ with } k \in \mathbb{Z}.$$

   Calculation :

$$(x-1)^2 = (2m-1)^2 = 4m^2 - 4m + 1 = 2(2m^2 - 2m) + 1.$$

   Thus $(x-1)^2 = 2k+1$ with $k = 2m^2 - 2m \in \mathbb{Z}$, i.e. odd. Therefore the contrapositive holds, and p is true.

Exercise 2 (4 pts) We define R on $\mathbb{R}^*$ by

$$xRy \iff x^2 - \frac{1}{x^2} = y^2 - \frac{1}{y^2}.$$

1. **Equivalence Relation Proof** To show that R is an equivalence relation, we must prove reflexivity, symmetry, and transitivity.
   Reflexivity : (0.5 pt) For any $x \in \mathbb{R}^*$,

   $$x^2 - \frac{1}{x^2} = x^2 - \frac{1}{x^2},$$

   hence xRx.
   Symmetry : (0.5 pt) Assume xRy, i.e.

   $$x^2 - \frac{1}{x^2} = y^2 - \frac{1}{y^2}.$$

   Then also

   $$y^2 - \frac{1}{y^2} = x^2 - \frac{1}{x^2},$$

   so yRx.
   Transitivity : (0.5 pt) Assume xRy and yRz, i.e.

   $$x^2 - \frac{1}{x^2} = y^2 - \frac{1}{y^2} \quad \wedge \quad y^2 - \frac{1}{y^2} = z^2 - \frac{1}{z^2}.$$

   Then

   $$x^2 - \frac{1}{x^2} = z^2 - \frac{1}{z^2},$$

   so xRz. Therefore R is reflexive, symmetric, and transitive, hence an equivalence relation.

2. **Equivalence class of 1 (0.5 pt) and deduction for -1 (0.5 pt)**. By the standard definition,

   $$[1] = \{\, y \in \mathbb{R}^* \mid 1Ry \,\}.$$

   Compute the condition :

   $$1Ry \iff 1^2 - \frac{1}{1^2} = y^2 - \frac{1}{y^2} \iff 0 = y^2 - \frac{1}{y^2} \iff y^4 = 1.$$

   Thus $y = \pm 1$ and

   $$[1] = \{1, -1\}.$$

   Since $-1 \in [1]$, we have $[-1] = [1] = \{1, -1\}$.

3. **Equivalence class of an arbitrary $x \in \mathbb{R}^*$ (1.5 pt)**. By definition :

   $$[x] = \{\, y \in \mathbb{R}^* \mid xRy \,\} = \{\, y \in \mathbb{R}^* \mid x^2 - \tfrac{1}{x^2} = y^2 - \tfrac{1}{y^2} \,\}.$$

   This equality gives

   $$(x^2 - y^2)(x^2y^2 + 1) = 0.$$

   Since $x^2y^2 + 1 > 0$, we must have $x^2 = y^2$, i.e. $y = \pm x$. Therefore

   $$[x] \underset{2}{=} \{x, -x\}.$$

Exercise 3 (8 pts)

Part I : We demonstrate that $(\mathbb{R}\setminus\{2\},*)$ forms a commutative group by verifying the group axioms. Let $x,y,z\in\mathbb{R}\setminus\{2\}$ and define

$$x*y=\frac{2x+2y-xy}{2}=x+y-\frac{xy}{2}.$$

(i) **Commutativity (0.5 pt)**

$$x*y=x+y-\frac{xy}{2}=y+x-\frac{yx}{2}=y*x.$$

(ii) **Associativity (0.5 pt)** Verify $(x*y)*z=x*(y*z)$:

$$(x*y)*z=\left(x+y-\frac{xy}{2}\right)+z-\frac{\left(x+y-\frac{xy}{2}\right)z}{2}=x+y+z-\frac{xy+xz+yz}{2}+\frac{xyz}{4},$$

and similarly

$$x*(y*z)=x+\left(y+z-\frac{yz}{2}\right)-\frac{x\left(y+z-\frac{yz}{2}\right)}{2}=x+y+z-\frac{xy+xz+yz}{2}+\frac{xyz}{4}.$$

Thus, $*$ is associative.

(iii) **Identity Element (0.5 pt)** Solve $x*e=x$:

$$x+e-\frac{xe}{2}=x\quad\Rightarrow\quad e\left(1-\frac{x}{2}\right)=0\quad\Rightarrow\quad e=0.$$

(iv) **Inverse Element (0.5 pt)** For each $x\neq 2$, solve $x*x^{-1}=0$:

$$x+x^{-1}-\frac{xx^{-1}}{2}=0\quad\Rightarrow\quad x^{-1}=\frac{-2x}{2-x}\quad(\text{valid since } x^{-1}\neq 2).$$

Conclusion : $(\mathbb{R}\setminus\{2\},*)$ is a commutative group.

Part II : Let $f:(\mathbb{R}\setminus\{2\},*)\to(\mathbb{R}^*,\times)$ be defined by

$$f(x)=\frac{2-x}{2}.$$

1. **Kernel and Image of f**

   - **Kernel (1 pt)** :

   $$\mathrm{Ker}(f)=\{x\mid \tfrac{2-x}{2}=1\}=\{0\}.$$

   - **Image (1 pt)** : For any $y\in\mathbb{R}^*$, solve $y=\frac{2-x}{2}\Rightarrow x=2(1-y)\neq 2$. Hence $\mathrm{Im}(f)=\mathbb{R}^*$.

2. **Homomorphism (1 pt)**. Verify $f(x*y)=f(x)\times f(y)$:

$$f(x*y)=\frac{2-\left(x+y-\frac{xy}{2}\right)}{2}=\left(\frac{2-x}{2}\right)\times\left(\frac{2-y}{2}\right)=f(x)\times f(y).$$

3. **Isomorphism**

   - **Injective (0.5 pt)** : $\mathrm{Ker}(f)=\{0\}$ implies f is injective.

- **Surjective (0.5 pt)** : $\mathrm{Im}(f) = \mathbb{R}^*$ implies f is surjective.

Conclusion : Thus f is bijective and homomorphism then is an isomorphism (**1 pt**).

4. **Additional Computations**

- **Image of** $[-2, 0]$ (**0.5 pt**) :

$$f([-2,0]) = \{f(x) \in \mathbb{R} \mid -2 \leq x \leq 0\}, \quad f(x) = \frac{2-x}{2}.$$

Since f is decreasing on $[-2, 0] \Rightarrow f(0) \leq f(x) \leq f(-2)$

$$\Rightarrow f([-2,0]) = [1,2].$$

- **Inverse Image of** $\{1\}$ (**0.5 pt**) :

$$f^{-1}(\{1\}) = \mathrm{Ker}(f) = \{x \mid \tfrac{2-x}{2} = 1\} = \{0\}.$$

Exercise 4 (3 pts)

Let R be an equivalence relation on a set E.

1. Since $\mathcal{R}$ is an equivalence relation then it is reflexive. Thus, $a\mathcal{R}a$ for all $a \in E$. Hence $a \in [a]$ (**1 pt**) .

2. **Prove** $a\mathcal{R}b \Leftrightarrow [a] = [b]$

i) (1 pt) $a\mathcal{R}b \Rightarrow [a] = [b]$. Assume $a\mathcal{R}b$. To show $[a] \subseteq [b]$, let $x \in [a]$, i.e. $x\mathcal{R}a$. By transitivity with $a\mathcal{R}b$, we get $x\mathcal{R}b$, hence $x \in [b]$. Thus $[a] \subseteq [b]$. Similarly, let $y \in [b]$, i.e. $y\mathcal{R}b$. Using symmetry ($a\mathcal{R}b \Rightarrow b\mathcal{R}a$) and transitivity, we obtain $y\mathcal{R}a$, hence $y \in [a]$. Thus $[b] \subseteq [a]$. Therefore $[a] = [b]$.

ii) (1 pt) $[a] = [b] \Rightarrow a\mathcal{R}b$. Suppose $[a] = [b]$. Since $a \in [a] = [b]$ (reflexivity), we have $a \in [b]$ $\Rightarrow a\mathcal{R}b$.