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First year in Computer Science 2025-2026

Course of Algebra 1

Chapter 3

Binary Relations

3.1 Definition

Definition 1. *Let E and F be two sets. A* relation *R between elements of E and F is a subset of the Cartesian product $E \times F$. We write xRy to indicate that the ordered pair (x, y) belongs to the relation R, i.e.:*

$$xRy \iff (x, y) \in R$$

If $E = F$, the relation R is called a binary relation *on E.*

3.2 Examples

Example 3.2.1.

- *Let $x, y \in \mathbb{N}$. Define xRy if and only if x divides y. This is a binary relation on $\mathbb{N}$.*
- *Let $x, y \in \mathbb{R}$. Define xRy if and only if $x \leq y$. This is a binary relation on $\mathbb{R}$.*
- *Let $A \subseteq E$, $B \subseteq F$. Define xRy if and only if $x \in A$ and $y \in B$. Then $R \subseteq A \times B$.*

3.3 Properties of Binary Relations

Let $R \subseteq E \times E$ be a binary relation and let $x, y, z \in E$. We say that R is:

- **Reflexive** if $\forall x \in E,\ xRx$
- **Symmetric** if $\forall x, y \in E,\ xRy \Rightarrow yRx$
- **Antisymmetric** if $\forall x, y \in E,\ xRy \wedge yRx \Rightarrow x = y$
- **Transitive** if $\forall x, y, z \in E,\ xRy \wedge yRz \Rightarrow xRz$

3.4 Equivalence and Order Relations

3.4.1 Equivalence Relation

Definition 2. *A binary relation $R \subseteq E \times E$ is called an* equivalence relation *if it satisfies the following three properties:*

- *Reflexive:* $\forall x \in E,\ xRx$
- *Symmetric:* $\forall x, y \in E,\ xRy \Rightarrow yRx$
- *Transitive:* $\forall x, y, z \in E,\ xRy \land yRz \Rightarrow xRz$

Example 3.4.1. *Let R be defined on E by $xRy \iff x = y$. This relation is reflexive, symmetric, and transitive, hence an equivalence relation.*

3.4.2 Order Relation

Definition 3. *A binary relation $R \subseteq E \times E$ is called an* order relation *if it satisfies:*

- *Reflexive:* $\forall x \in E,\ xRx$
- *Antisymmetric:* $\forall x, y \in E,\ xRy \land yRx \Rightarrow x = y$
- *Transitive:* $\forall x, y, z \in E,\ xRy \land yRz \Rightarrow xRz$

Example 3.4.2. *Let $A, B \subseteq E$. Define $ARB \iff A \subseteq B$. This relation is reflexive, antisymmetric, and transitive, so it is an order relation.*

Let $x, y \in \mathbb{R}$. Define $xRy \iff x \leq y$. This is also an order relation.

3.4.3 Total and Partial Order

Definition 4. *An order relation R on a set E is called a* total order *if every pair of elements is comparable:*

$$\forall x, y \in E,\ xRy \text{ or } yRx$$

Otherwise, R is called a partial order.

Example 3.4.3.

- *The relation $x \leq y$ on $\mathbb{R}$ is a total order.*
- *The relation x divides y on $\{0, 1, 2, 3\}$ is a partial order: not all pairs are comparable.*

3.5 Equivalence Class

Definition 5. *Let R be an equivalence relation on a set E, and let $x \in E$. The* equivalence class *of x, denoted $[x]$ or C_x, is the set of all elements in E that are related to x under R:*

$$[x] = \{y \in E \mid xRy\}$$

3.5.1 Properties of Equivalence Classes

Let R be an equivalence relation on a set E. For any $x, y \in E$, the equivalence classes satisfy:

- $x \in [x]$ (every element belongs to its own class)
- If xRy, then $[x] = [y]$
- If $\neg(xRy)$, then $[x] \cap [y] = \emptyset$

Remark 3.5.1. *The set of all equivalence classes forms a partition of E. That is, the classes are disjoint and their union equals E:*

$$E = \bigcup_{x \in E} [x]$$

3.5.2 Quotient Set

Definition 6. *Let R be an equivalence relation on a set E. The set of all equivalence classes under R is called the* quotient set *of E by R, denoted:*

$$E/R = \{[x] \mid x \in E\}$$

3.5.3 Example

Let R be a relation on $\mathbb{R}$ defined by:

$$xRy \iff x - y \in \mathbb{Z}$$

We verify that R is an equivalence relation:

- Reflexive: $x - x = 0 \in \mathbb{Z} \Rightarrow xRx$
- Symmetric: If $x - y \in \mathbb{Z}$, then $y - x = -(x - y) \in \mathbb{Z} \Rightarrow yRx$
- Transitive: If $x - y \in \mathbb{Z}$ and $y - z \in \mathbb{Z}$, then $(x - y) + (y - z) = x - z \in \mathbb{Z} \Rightarrow xRz$

Now we compute some equivalence classes:

- $[0] = \{x \in \mathbb{R} \mid x - 0 \in \mathbb{Z}\} = \mathbb{Z}$
- $[1] = \{x \in \mathbb{R} \mid x - 1 \in \mathbb{Z}\} = \mathbb{Z} + 1$
- $[\frac{1}{2}] = \left\{ x \in \mathbb{R} \mid x - \frac{1}{2} \in \mathbb{Z} \right\} = \mathbb{Z} + \frac{1}{2}$

Remark 3.5.2. *For example, $\frac{1}{4} \notin [\frac{1}{2}]$ since $\frac{1}{4} - \frac{1}{2} = -\frac{1}{4} \notin \mathbb{Z}$. Thus, $[\frac{1}{4}] \neq [\frac{1}{2}]$.*