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Course of Algebra 1

Chapter 2

Set Theory and Mappings

2.1 Algebra of Sets

2.1.1 Set

Definition 1. *A* set *is a well-defined collection of distinct objects, called* elements. *We write $x \in A$ if x is an element of the set A.*

Example 2.1.1. *1. E: the set of students at M'sila University.*

2. $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$: the set of natural numbers.

3. $E = \{x \in \mathbb{N} \mid x \text{ is divisible by } 2\}$: the set of even natural numbers.

4. $\emptyset$: the empty set, containing no elements.

2.1.2 Types of Sets

Definition 2. *Sets can be classified according to their size or relationship with other sets.*

Example 2.1.2.
- ***Finite set:*** *A set with a limited number of elements, e.g. $A = \{1, 2, 3, 4\}$.*
- ***Infinite set:*** *A set with infinitely many elements, e.g. $\mathbb{N}$.*
- ***Singleton set:*** *A set with exactly one element, e.g. $\{a\}$.*
- ***Universal set:*** *The set containing all objects under consideration, usually denoted E.*

2.1.3 Inclusion

Definition 3. *A set A is a* subset *of a set B, denoted $A \subseteq B$, if every element of A is also an element of B.*

$$A \subseteq B \iff \forall x \in A,\ x \in B$$

Definition 4 (Proper Subset)**.** *If $A \subseteq B$ and $A \neq B$, then A is called a* proper subset *of B, denoted $A \subset B$. This means A is contained in B but does not equal B.*

Remark 2.1.3. *The negation is written:*

$$A \not\subseteq B \iff \exists x \in A \text{ such that } x \notin B$$

Example 2.1.4.
- *$\mathbb{N} \subseteq \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers.*
- *$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.*

2.1.4 Equality of Sets

Definition 5. *Two sets A and B are equal, denoted $A = B$, if they contain exactly the same elements.*

$$A = B \iff (A \subseteq B \text{ and } B \subseteq A)$$

2.1.5 Set Difference

Definition 6. *The* difference *of two sets A and B, denoted $A \setminus B$ or $A - B$, is the set of elements that belong to A but not to B.*

$$A \setminus B = \{x, \quad x \in A \wedge x \notin B\}$$

Definition 7. *Let $A \subseteq E$, where E is a universal set. The* complement *of A in E is the set of all elements of E not in A.*

$$C_E(A) = C_E^A = E \setminus A = \{x \in E \mid x \notin A\}$$

Remark 2.1.5. *Other common notations for the complement of A include:*

$$A^c, \quad \overline{A}, \quad A', \quad E - A$$

2.2 Set Operations

2.2.1 Union

Definition 8. *Let A and B be two subsets of a universal set E. The* union *of A and B, denoted $A \cup B$, is the set of all elements that belong to A, to B, or to both:*

$$A \cup B = \{x \in E, x \in A \text{ or } x \in B\}$$

Remark 2.2.1. *Logical characterization:*

$$x \in A \cup B \iff x \in A \vee x \in B$$

$$x \notin A \cup B \iff x \notin A \wedge x \notin B$$

2.2.2 Intersection

Definition 9. *Let A and B be two subsets of a universal set E. The* intersection *of A and B, denoted $A \cap B$, is the set of all elements that belong to both A and B:*

$$A \cap B = \{x \in E, x \in A \text{ and } x \in B\}$$

Remark 2.2.2. *Logical characterization:*

$$x \in A \cap B \iff x \in A \wedge x \in B$$

$$x \notin A \cap B \iff x \notin A \vee x \notin B$$

Remark 2.2.3. *1. If $A \cap B = \emptyset$, then A and B are said to be* disjoint*.*

2. If $B = C_1 \cup C_2$, then:

$$B \setminus A = (C_1 \setminus A) \cup (C_2 \setminus A)$$

3. The difference can be expressed using complement:

$$A \setminus B = A \cap B^c$$

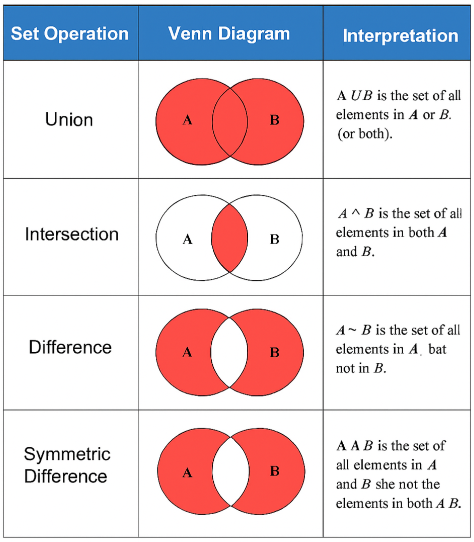
2.2.3 Symmetric Difference

Definition 10. *Let A and B be two subsets of a universal set E. The* symmetric difference *of A and B, denoted $A \triangle B$, is the set of elements that belong to either A or B, but not to both:*

$$A \triangle B = (A \setminus B) \cup (B \setminus A)$$

Remark 2.2.4. *Alternative expression using complements:*

$$A \triangle B = (A \cup B) \setminus (A \cap B)$$



Set Operation	Venn Diagram	Interpretation
Union	A B	A *UB* is the set of all elements in ***A*** or *B*. (or both).
Intersection	A B	*A* ^ *B* is the set of all elements in both ***A*** and *B*.
Difference	A B	*A* ~ *B* is the set of all elements in ***A***. bat not in *B*.
Symmetric Difference	A B	A A *B* is the set of all elements in *A* and *B* she not the elements in both *A B*.

Set Operations

2.3 Properties of Set Operations

2.3.1 Commutativity

Definition 11. *Let A and B be subsets of a universal set E. The operations of union and intersection are* commutative*, meaning:*

$$A \cup B = B \cup A, \quad A \cap B = B \cap A$$

2.3.2 Associativity

Definition 12. *For any sets $A, B, C \subseteq E$, the operations of union and intersection are* associative*:*

$$A \cup (B \cup C) = (A \cup B) \cup C, \quad A \cap (B \cap C) = (A \cap B) \cap C$$

2.3.3 Distributivity

Definition 13. *For any sets $A, B, C \subseteq E$, union and intersection satisfy the* distributive laws*:*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

2.3.4 Idempotence

Definition 14. *For any set $A \subseteq E$, the operations are* idempotent*:*

$$A \cup A = A, \quad A \cap A = A$$

2.3.5 De Morgan's Laws

Definition 15. *Let $A, B \subseteq E$. The* De Morgan Laws *relate complements of unions and intersections:*

$$(A \cup B)^c = A^c \cap B^c, \quad (A \cap B)^c = A^c \cup B^c$$

Remark 2.3.1. *Proof of the first law:*

$$x \in (A \cup B)^c \iff x \notin A \cup B \iff x \notin A \wedge x \notin B \iff x \in A^c \cap B^c$$

Thus, $(A \cup B)^c = A^c \cap B^c$. The second law follows by similar reasoning:

$$x \in (A \cap B)^c \iff x \notin A \cap B \iff x \notin A \vee x \notin B \iff x \in A^c \cup B^c$$

Therefore, $(A \cap B)^c = A^c \cup B^c$.

2.4 Power Set and Cardinality

2.4.1 Power Set

Definition 16. *Let E be a set. The* power set *of E, denoted $\mathcal{P}(E)$, is the set of all subsets of E, including the empty set and E itself:*

$$\mathcal{P}(E) = \{A \subseteq E\}$$

Remark 2.4.1. *If $E \neq \emptyset$, then $\mathcal{P}(E) \neq \emptyset$. In fact, $\mathcal{P}(E)$ always contains at least the empty set: $\emptyset \in \mathcal{P}(E)$.*

Example 2.4.2. *Let $E = \{a, b\}$. Then:*

$$\mathcal{P}(E) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

2.4.2 Cardinality

Definition 17. *The* cardinality *of a finite set A, denoted $Card(A)$, is the number of elements in A.*

Remark 2.4.3. *If A is a finite set with n elements, then:*

$$Card(\mathcal{P}(A)) = 2^n$$

Example 2.4.4.

- *If $A = \{1, 2, 3\}$, then $Card(A) = 3$, and $Card(\mathcal{P}(A)) = 2^3 = 8$.*
- *If $B = \{x, y\}$, then $Card(B) = 2$, and $\mathcal{P}(B) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}$.*

2.5 Mappings (Applications)

2.5.1 Definition

Definition 18. *Let E and F be two sets. A* mapping *(or* application*) from E to F is a rule that assigns to each element $x \in E$ a unique element $y \in F$. This mapping is denoted by:*

$$f : E \to F, \quad x \mapsto f(x)$$

Here, E is called the domain*, F the* codomain*, and $f(x)$ the* image *of x under f.*

2.5.2 Examples

Example 2.5.1.

- $f_1 : \mathbb{R} \to \mathbb{R}, \quad f_1(x) = 4x + 2$
- $f_2 : \mathbb{R} \to \mathbb{R}, \quad f_2(x) = 5x + 3$

2.5.3 Direct Image

Definition 19. *Let $f : E \to F$ be a mapping and let $A \subseteq E$. The* direct image *of A under f, denoted $f(A)$, is the subset of F defined by:*

$$f(A) = \{f(x) \in F, x \in A\}$$

2.5.4 Inverse Image

Definition 20. *Let $f : E \to F$ be a mapping and let $B \subseteq F$. The* inverse image *of B under f, denoted $f^{-1}(B)$, is the subset of E defined by:*

$$f^{-1}(B) = \{x \in E, f(x) \in B\}$$

2.5.5 Examples

Example 2.5.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x + 1$.*

- $f([0, 1]) = [1, 3]$
- $f(\{0\}) = \{1\}$
- $f(\{0, 1\}) = \{1, 3\}$
- $f((0, 1)) = (1, 3)$

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = (2x - 1)^2$.

- $f^{-1}([0, 1]) = [0, 1]$
- $f^{-1}((0, 1)) = (0, 1)$

2.5.6 Injection (One-to-One)

Definition 21. *Let $f : E \to F$ be a mapping. We say that f is* injective *(or* one-to-one*) if distinct elements of E have distinct images in F. Formally:*

$$f \text{ is injective } \iff \forall x_1, x_2 \in E,\ f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

Equivalently:

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$

Example 2.5.3.

- $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = 2x + 1$ *is injective, since* $f(x_1) = f(x_2) \Rightarrow 2x_1 + 1 = 2x_2 + 1 \Rightarrow x_1 = x_2$.
- $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2$ *is not injective, since* $f(-1) = f(1) = 1$ *but* $-1 \neq 1$.

2.5.7 Surjection (Onto)

Definition 22. *Let $f : E \to F$ be a mapping. We say that f is* surjective *(or* onto*) if every element of F is the image of at least one element of E. Formally:*

$$f \text{ is surjective } \iff \forall y \in F, \exists x \in E \text{ such that } f(x) = y$$

Equivalently, $f(E) = F$.

Example 2.5.4.

- $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2$ *is not surjective, since negative numbers have no preimage.*
- $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^3$ *is surjective, since every* $y \in \mathbb{R}$ *has a preimage* $x = \sqrt[3]{y}$.

2.5.8 Bijective Mapping (One-to-One Correspondence)

Definition 23. *Let $f : E \to F$ be a mapping. We say that f is* bijective *(or a* one-to-one correspondence*) if it is both injective and surjective. This means that every element $y \in F$ is the image of a unique element $x \in E$. Formally:*

$$f \text{ is bijective } \iff \forall y \in F, \exists! x \in E \text{ such that } f(x) = y$$

Remark 2.5.5. *If f is bijective, then the inverse function $f^{-1} : F \to E$ exists and is also bijective.*

Example 2.5.6. *Let* $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = 5x + 3$. *This function is bijective because:*

- *It is injective:* $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
- *It is surjective: for every* $y \in \mathbb{R}$, *there exists* $x = \frac{y-3}{5} \in \mathbb{R}$ *such that* $f(x) = y$

2.5.9 Composition of Functions

Definition 24. *Let E, F, G be sets, and let $f : E \to F$, $g : F \to G$ be two functions. The* composition *of f and g, denoted $g \circ f$, is the function from E to G defined by:*

$$(g \circ f)(x) = g(f(x)), \quad \text{for all } x \in E$$

Remark 2.5.7.

- *If f and g are injective, then $g \circ f$ is injective.*
- *If f and g are surjective, then $g \circ f$ is surjective.*
- *If f and g are bijective, then $g \circ f$ is bijective.*
- *In particular, if $f : E \to F$ is bijective with inverse $f^{-1} : F \to E$, then:*

$$f \circ f^{-1} = Id_F, \quad f^{-1} \circ f = Id_E$$

2.5.10 Properties of Functions

Let $f : E \to F$ be a function and let $A, B \subseteq E$. Then:

- If $A \subseteq B$, then $f(A) \subseteq f(B)$
- $f(A \cup B) = f(A) \cup f(B)$
- $f(A \cap B) \subseteq f(A) \cap f(B)$
- $f(A \setminus B) \supseteq f(A) \setminus f(B)$

Let $C, D \subseteq F$. Then:

- $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$
- $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$
- $f^{-1}(C \setminus D) = f^{-1}(C) \setminus f^{-1}(D)$

Remark 2.5.8. *For example, to prove $f(A \cap B) \subseteq f(A) \cap f(B)$: Let $x \in f(A \cap B)$. Then there exists $a \in A \cap B$ such that $f(a) = x$. Since $a \in A$ and $a \in B$, we have $x \in f(A)$ and $x \in f(B)$, so $x \in f(A) \cap f(B)$.*

Proposition 2.5.9. *Let $f : E \to F$ and $g : F \to G$ be functions. Then:*

- *If $g \circ f$ is injective, then f is injective.*
- *If $g \circ f$ is surjective, then g is surjective.*
- *If $g \circ f$ is bijective, then f is injective and g is surjective.*

Proof. **(1)** Suppose $g \circ f$ is injective. Let $x, x' \in E$ such that $f(x) = f(x')$. Then $g(f(x)) = g(f(x'))$, so $(g \circ f)(x) = (g \circ f)(x')$. By injectivity of $g \circ f$, we conclude $x = x'$. Thus, f is injective.

(2) Suppose $g \circ f$ is surjective. Let $y \in G$. Then there exists $x \in E$ such that $g(f(x)) = y$. Let $z = f(x) \in F$. Then $g(z) = y$, so every $y \in G$ has a preimage in F. Thus, g is surjective.

(3) Suppose $g \circ f$ is bijective. Then it is both injective and surjective. From (1), f is injective. From (2), g is surjective. □