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Course of Algebra 1

Chapter 1

Logic

1.1 Elements of Logic and Reasoning Methods

Are the following assertions true or false? How can we get a certain judgment?

- Example: $x \geq 1$? (This is incorrect mathematically)
- All that he is saying is not true?
- Imagine you are driving without traffic lights.
- How can you know the time of prayer?
- Let a group of persons be on a circle. Who is in the first position?

To avoid such anomalies, mathematicians use very precise language, which allows ascertaining logical facts.

So, Logic is the science of arguments validity and demonstration rigor, ensures the certainty of our statements (Propositions)

1.2 Rules of Formal Logic

Definition 1. *A **proposition** is a statement or mathematical expression that is either definitely true or definitely false, but not both.*

Notation: We use letters to represent propositions: $p, q, r, \ldots$

Example 1.2.1. *Examples of propositions:*

- *"The square root of 2 is irrational" - True proposition*
- *"$2 + 2 = 5$" - False proposition*
- *"Algiers is the capital of Algeria" - True proposition*
- *"Riyadh is not the capital of Saudi Arabia" - False proposition*

Example 1.2.2. *Non-propositions (cannot be assigned truth values):*

- *$x \geq 0$ (not a proposition, because we don't know the values of x)*
- *"You are an intelligent person" - Subjective, not a proposition*

- *"What time is it?" - Interrogative, not a proposition*
- *"Close the door!" - Imperative, not a proposition*

Definition 2. *Any proposition that has been proven true is called a* ***theorem*** *(for example, the Pythagorean theorem, Thales' theorem).*

Properties

- The statement or proposition taken to be true without proof is called an **axiom** (postulate)
- The special kind of axiom which introduces new concepts is called a **definition**
  - (e.g., A triangle is a polygon with 3 edges and 3 vertices.)
- The proposition that is considered auxiliary to proving another one, usually as a consequence
- The theorem that is usually a consequence of another theorem is called a **corollary**

Remark 1.2.3. *The statements that are neither true nor false are called* ***non-propositions****.*

1.3 Truth Tables

Definition 3. *A* ***truth table*** *is a mathematical representation that displays all possible truth values of logical expressions. It consists of rows and columns that show the truth values for all possible combinations of the variables involved, we write* ***'T'*** *for True and* ***'F'*** *for False.*

Truth table for a single proposition p:

p
T
F

Remark 1.3.1. *The number of cases in a truth table is given by 2^n, where n is the number of propositional variables. For binary truth values (T/F), we have 2^n possible combinations.*

1.4 Logical Connectives

Definition 4. *Logical operators (connectives) are symbols that combine propositions to form compound statements.*

1.4.1 Negation Operator

Definition 5. *Let p be a proposition. The* ***negation*** *of p is a proposition that denotes the opposite truth value, denoted as $\bar{p}$ (read as "not p").*

Truth table for negation:

p	$\bar{p}$
T	F
F	T

Example 1.4.1. *Double negation truth table:*

p	$\bar{p}$	$\bar{\bar{p}}$
T	F	T
F	T	F

- p: "the function f is positive", then $\bar{p}$: "the function f is not positive"
- p: "$x+3=0$", then $\bar{p}$: "$x+3 \neq 0$"

Now, let p, q be two propositions.

1.4.2 Conjunction Operator

Definition 6. *The **conjunction** is the logical connection (AND) of two propositions, denoted as (p and q) or ($p \wedge q$). The conjunction $p \wedge q$ is true only when both p and q are true.*

Truth table for conjunction:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 1.4.2.
- *"2 is even and 3 is a prime number" - This proposition is true*
- *"$1 \leq 0$ and $4 \geq 2$" - This proposition is false*

1.4.3 Disjunction Operator

Definition 7. *The **disjunction** is a logical connection denoted by $\vee$. We denote the disjunction between p and q as ($p \vee q$). The disjunction $p \vee q$ is false only when both p and q are false, otherwise it is true.*

Truth table for disjunction:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 1.4.3.
- *"2 is odd or 3 is a prime number" - True*
- *"For $x \in \mathbb{R}$, $x^2 < 0$ or $3 \leq 2$" - False*

1.4.4 Implication Operator

Definition 8. *The **implication** of two propositions p, q is denoted as $p \Rightarrow q$, which means "p implies q" or "if p then q". The implication $p \Rightarrow q$ is false if p is true and q is false, otherwise it is true for all other cases.*

Truth table for implication:

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Example 1.4.4.

- *"$0 \leq x \leq 9 \implies 4x \leq 36$" - True (corrected from $4.(x) \leq 4.(9)$).*
- *"If a number is even then it is prime" - False (counterexample: 4 is even but not prime)*
- *If it's raining then the ground is wet - True*

Remark 1.4.5.

- *The **converse** of an implication $p \Rightarrow q$ is the implication $q \Rightarrow p$.*
- *The implication $p \Rightarrow q$ is equivalent to $\bar{p} \vee q$.*
- *The **contrapositive** of $p \Rightarrow q$ is $\bar{q} \Rightarrow \bar{p}$, which is equivalent to the original implication. i.e, $(p \Rightarrow q \Leftrightarrow \bar{q} \Rightarrow \bar{p})$.*
- *The **negation** of $p \Rightarrow q$ is $p \wedge \bar{q}$.*

Exercise:
Prove the forth property.

1.4.5 Equivalence Operator

Definition 9. *The equivalence of two propositions p and q is denoted $p \Leftrightarrow q$, which can also be written as $(p \Rightarrow q) \wedge (q \Rightarrow p)$. It can be read as "$p$ if and only if q" (or "p iff q").*

- *We say that $p \Leftrightarrow q$ if p and q have the same truth value, otherwise $(p \Leftrightarrow q)$ is false.*

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Example 1.4.6.

- $x \cdot 2 = 6 \Leftrightarrow x = 3$.
- *Sami won with his team of football $\not\Leftrightarrow$ Sami is a player of football. However, the first implication is true but the inverse is false.*

We summarize the following seven laws let p and q be two propositions.

- $\mathbf{p} \Leftrightarrow \mathbf{p}$ (Idempotence)
- $(\mathbf{p} \wedge \mathbf{q}) \Leftrightarrow (\mathbf{q} \wedge \mathbf{p})$ (Commutativity of $\wedge$)
- $(\mathbf{p} \vee \mathbf{q}) \Leftrightarrow (\mathbf{q} \vee \mathbf{p})$ (Commutativity of $\vee$)
- $((\mathbf{p} \wedge \mathbf{q}) \wedge \mathbf{r}) \Leftrightarrow (\mathbf{p} \wedge (\mathbf{q} \wedge \mathbf{r}))$ (Associativity of $\wedge$)
- $((\mathbf{p} \vee \mathbf{q}) \vee \mathbf{r}) \Leftrightarrow (\mathbf{p} \vee (\mathbf{q} \vee \mathbf{r}))$ (Associativity of $\vee$)
- $\mathbf{p} \vee (\mathbf{q} \wedge \mathbf{r}) \Leftrightarrow (\mathbf{p} \vee \mathbf{q}) \wedge (\mathbf{p} \vee \mathbf{r})$ (Distributivity of $\vee$ over $\wedge$)

- $\mathbf{p} \wedge (\mathbf{q} \vee \mathbf{r}) \Leftrightarrow (\mathbf{p} \wedge \mathbf{q}) \vee (\mathbf{p} \wedge \mathbf{r})$ (Distributivity of $\wedge$ over $\vee$)
- $\bar{\bar{\mathbf{p}}} \Leftrightarrow \mathbf{p}$ (Double Negation Law)
- $\overline{\mathbf{p} \wedge \mathbf{q}} \Leftrightarrow \bar{\mathbf{p}} \vee \bar{\mathbf{q}}$ (De Morgan's Laws)
- $\overline{\mathbf{p} \vee \mathbf{q}} \Leftrightarrow \bar{\mathbf{p}} \wedge \bar{\mathbf{q}}$ (De Morgan's Laws)
- $\mathbf{p} \implies \mathbf{q} \Leftrightarrow \bar{\mathbf{p}} \vee \mathbf{q} \Leftrightarrow \bar{\mathbf{q}} \implies \bar{\mathbf{p}}$ (Conditional Law and Contrapositive)

1.5 Quantifiers

Universal Quantifier $\forall$: "for all"

- The statement "for all $x, P(x)$" is denoted as $(\forall \mathbf{x})\mathbf{P}(\mathbf{x})$, where $P(x)$ is a predicate (a statement that depends on the variable x).
- We say that $(\forall x)P(x)$ is true if $P(x)$ is true for all x in the domain.

Remark 1.5.1. *Here, $P(x)$ represents a* ***predicate*** *or* ***propositional function*** *that becomes a proposition when x is assigned a specific value from the domain.*

Example 1.5.2. *Examples of $P(x)$:*

- *$P(x)$: "x is greater than 0" where $x \in \mathbb{R}$*
- *$P(x)$: "x is a prime number" where $x \in \mathbb{N}$*
- *$P(x)$: "x is a student at University of M'sila"*

Existential Quantifier $\exists$: "there exists"

- The statement "there exists x such that $P(x)$" is denoted as $(\exists \mathbf{x})\mathbf{P}(\mathbf{x})$.

The statement "there exists a unique x in the set E such that $P(x)$" is denoted as $(\exists ! \mathbf{x} \in \mathbf{E})\mathbf{P}(\mathbf{x})$.

Example 1.5.3. ***Express the following statements using quantifiers:***

1. *The function f is zero for all real numbers x $(\forall x \in \mathbb{R})f(x) = 0$.*
2. *The function f is zero for at least one real number $(\exists x \in \mathbb{R})f(x) = 0$.*

Remark 1.5.4. *The statements $(\forall x)(\exists y)P(x, y)$ and $(\exists y)(\forall x)P(x, y)$ are different. In the first one, y depends on x, while in the second one, y does not depend on x.*

Example 1.5.5.
- *$(\forall x \in \mathbb{R})(\exists y \in \mathbb{R}) : x + y \geq 0$ is true. In fact, $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R}) : y = |x| + 4$ is a valid choice.*
- *$(\exists y \in \mathbb{R})(\forall x \in \mathbb{R}) : x + y \geq 0$ This is* ***false****, because $(\forall x \in \mathbb{R})$ the value of y must be $\geq -x$, which isn't possible for a fixed y as x is not restricted.*

Rules of Negation

Let $P(x)$ be a proposition.

1. $\overline{\forall x \in E, P(x)}$ is: $\exists x \in E, \overline{P(x)}$.

2. $\overline{\exists x \in E, P(x)}$ is: $\forall x \in E, \overline{P(x)}$.

Remark 1.5.6.
- $(\exists x \in E)(\forall y \in E), P(x, y)$ *means that* x *is a* ***constant (fixed)****, independent of* y*, which* y *varies in* E.
- $(\forall x \in E)(\exists y \in E), P(x, y)$ *means that* y *depends on* x *through a certain relation* R *such that* $y = R(x)$.
- $\overline{(\forall x)(\forall y)P(x, y)} \Leftrightarrow (\exists x)(\exists y)\overline{P(x, y)}$.
- $\overline{(\exists x)(\exists y)P(x, y)} \Leftrightarrow (\forall x)(\forall y)\overline{P(x, y)}$.
- $\overline{(\forall x)(\exists y)P(x, y)} \Leftrightarrow (\exists x)(\forall y)\overline{P(x, y)}$.
- $\overline{(\exists x)(\forall y)P(x, y)} \Leftrightarrow (\forall x)(\exists y)\overline{P(x, y)}$.

Example 1.5.7.
- $\overline{(\forall \epsilon > 0, \exists a \in \mathbb{Q}_+ \textit{ such that } 0 < a < \epsilon)} \Leftrightarrow \exists \epsilon > 0,\ \forall a \in \mathbb{Q}_+,\ a \leq 0$ *or* $a \geq \epsilon$

2. The Methods of Reasoning

What is Reasoning? It is the strategy you use to answer a question and start a demonstration. There are several types of reasoning.

- To show that $(p \implies q)$ is true we can use:

2.1 Direct Reasoning Method

Assume that p is true and prove that q is also true.

Example 2.1.1 Let's prove that for $n \in \mathbb{N}$, if n is even then n^2 is even.

Proof.
- Assume that n is even, i.e., $\exists k \in \mathbb{Z}$ such that $n = 2k$.
- Therefore, $n^2 = (2k)^2 = 2(2k^2)$.
- Put $2k^2 = k'$, so $n^2 = 2k'$, thus $\exists k' \in \mathbb{Z}$ such that $n^2 = 2k'$.
- Which means that n^2 is even.

$\square$

2.2 Contrapositive Reasoning Method

Since $(p \implies q) \Leftrightarrow (\bar{q} \implies \bar{p})$, to prove $(p \implies q)$, we use the contrapositive $(\bar{q} \implies \bar{p})$. Assume that $\bar{q}$ is true and prove that $\bar{p}$ is true.

Example 2.2.1

- Prove that for $n \in \mathbb{N}$, if n^2 is even, then n is even.

Proof.

- By contrapositive, it suffices to show that if n is odd, then n^2 is odd.
- This is an exercise (as shown in the previous example). Hence the result is proven.

□

2.3 Proof by Contradiction

To show that r is a true proposition, we assume that $\bar{r}$ is true and we arrive at a contradiction (absurd).

- When r is $(p \implies q)$, by contradiction, we assume that $p \wedge \bar{q}$ is true and we get a contradiction.

Example 2.3.1 (Continuation of Proof by Contradiction)

- **Theorem:** For all $a, b > 0$, if $a^2 + b^2 = 2ab$, then $a = b$.
- **Proof:** By contradiction, we assume that $a \neq b$ and $a, b > 0$.
- Since $a^2 + b^2 = 2ab$, we have $a^2 - 2ab + b^2 = 0$.
- Thus, $(a - b)^2 = 0$, which means $a - b = 0$.
- Since $a - b = 0$, we have $a = b$.
- This is a contradiction ($\mathbf{a} \neq \mathbf{b}$ and $\mathbf{a} = \mathbf{b}$).
- We conclude that for all $a, b > 0$, $a^2 + b^2 = 2ab \implies a = b$ is true.

2.4 Counterexample Method

To show that a proposition is false, it is sufficient to provide a counterexample, which means a specific case for which the proposition is false.

Example 2.4.1

- **Proposition:** If n is even, then $n^2 + 1$ is even.
- **Counterexample:** For $n = 2$, $n^2 + 1 = 5$ is not even.

2.5 Proof by Induction

To show that $P(n)$ is true for all $n \in \mathbb{N}$ (or for all $n \geq n_0$, where n_0 is the initial term), we follow these steps:

1. We show that $P(n_0)$ is true (Initial case).
2. We assume that $P(n)$ is true at order n (Induction Hypothesis).
3. We show that $P(n+1)$ is true at order $n+1$. Then $P(n)$ is true for all n.

Example 2.5.1

- **Show:** $\forall n \in \mathbb{N}^*$, $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.