M'sila Uni vercity 2024/2025

Algebra1 Serie number3

Exercise 1. In $\mathbb{R}$, the binary relation $\mathcal{R}$ is defined as follows:

$$\forall x, y \in \mathbb{R} : x\mathcal{R}y \iff x^2 - 1 = y^2 - 1$$

1. Show that $\mathcal{R}$ is an equivalence relation on $\mathbb{R}$.

2. Determine the quotient set $\mathbb{R}/\mathcal{R}$.

Exercise 2. For every $n \in \mathbb{N}^*$, a binary relation on $\mathbb{Z}$ is defined by

$$\forall x, y \in \mathbb{Z} : x\mathcal{R}y \iff \exists k \in \mathbb{Z} \mid x - y = kn$$

1. Show that $\mathcal{R}$ is an equivalence relation on $\mathbb{Z}$.

2. Assume that $n = 3$:

(a) Determine the equivalence class of $x \in \mathbb{Z}$. Deduce the classes $\overline{0}, \overline{1}, \overline{2}$.

(b) Show that $\forall m \in \mathbb{Z} : \overline{0} = \overline{3m}, \overline{1} = \overline{3m+1}, \overline{2} = \overline{3m+2}$.

(c) Show that $\overline{0} \cap \overline{1} = \emptyset, \overline{1} \cap \overline{2} = \emptyset, \overline{0} \cap \overline{2} = \emptyset$. Deduce the quotient set $\mathbb{Z}/\mathcal{R}$.

Exercise 3. Let E be a set and let A be a subset of E. A binary relation $\mathcal{R}$ is defined on $\mathcal{P}(E)$ as follows:

$$\forall X, Y \in \mathcal{P}(E) : X\mathcal{R}Y \iff A \cap X = A \cap Y$$

1. Show that $\mathcal{R}$ is an equivalence relation on $\mathcal{P}(E)$.

2. Determine the equivalence classes of $\emptyset$ and E. Deduce $\bar{A}$ and $\overline{C_E(A)}$.

Exercise 4 A binary relation $\mathcal{R}$ is defined on $\mathbb{R}^2$ as follows:

$\forall (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2 : (x_1, y_1) \mathcal{R} (x_2, y_2) \iff x_1 \leq x_2$ and $y_1 \leq y_2$.

1. Show that $\mathcal{R}$ is an order relation on $\mathbb{R}^2$.

2. Are the elements $(2, 4), (3, 1)$ of $\mathbb{R}^2$ comparable by $\mathcal{R}$?

3. Is the order total on $\mathbb{R}^2$?

4. Determine the set of upper bounds of $A = \{(1,2),(3,1)\} \subset \mathbb{R}^2$.

Exercise 1.

1. $\forall x, y \in \mathbb{R} : x\mathcal{R}y \iff x^2 - 1 = y^2 - 1$

(i) Reflexivity: $\forall x \in \mathbb{R}, x^2 - 1 = x^2 - 1 \Rightarrow xRx$.

(ii) Symmetry: $x\mathcal{R}y \Leftrightarrow x^2 - 1 = y^2 - 1 \Rightarrow y^2 - 1 = x^2 - 1 \Rightarrow y\mathcal{R}x$.

(iii) Transitivity:

$$\begin{cases} x\mathcal{R}y \\ y\mathcal{R}z \end{cases} \Leftrightarrow \begin{cases} x^2 - 1 = y^2 - 1 \\ y^2 - 1 = z^2 - 1 \end{cases} \Rightarrow x^2 - 1 = z^2 - 1 \Rightarrow x\mathcal{R}z.$$

Therefore, $\mathcal{R}$ is an equivalence relation.

2. $\mathbb{R}/\mathbb{R} = \{\bar{x} : x \in \mathbb{R}\}$.

We have $\bar{x} = \{y \in \mathbb{R} \mid y\mathbb{R}x\} = \{y \in \mathbb{R} \mid y^2 - 1 = x^2 - 1\} = \{x, -x \mid x \in \mathbb{R}\}$

Thus, $\mathbb{R}/\mathbb{R} = \{\{x_1 - x\}, x \in \mathbb{R}\}$.

Exercise 2.

1. $\forall x, y \in \mathbb{Z} : x\mathcal{R}y \Leftrightarrow \exists k \in \mathbb{Z} \mid x - y = kn.$

- Reflexivity: We know that $\forall x \in \mathbb{Z} : x - x = 0 = 0 \cdot n$ with $k = 0 \in \mathbb{Z}$, so $x\mathcal{R}x$.
- Symmetry: $x\mathcal{R}y \Leftrightarrow x - y = kn \Rightarrow y - x = (-k) \cdot n = k' \cdot n$ with $k' = -k \in \mathbb{Z}$. Thus, $y\mathcal{R}x$.

- Transitivity:

$$\begin{cases} x\mathcal{R}y \\ y\mathcal{R}z \end{cases} \Leftrightarrow \begin{cases} x - y = k_1 \cdot n / k_1 \in \mathbb{Z} \\ y - z = k_2 \cdot n / k_2 \in \mathbb{Z} \text{ gives:} \end{cases} \text{; Summing both sides:}$$

$x - z = (k_1 + k_2)n = k_3 \cdot n$ with $k_3 = k_1 + k_2 \in \mathbb{Z}$

Therefore, $x\mathcal{R}z$

2. For $n = 3 : \forall x, y \in \mathbb{Z} : x\mathcal{R}y \Leftrightarrow \exists k \in \mathbb{Z} : x - y = 3k.$

(a) For any

$$x \in \mathbb{Z} : \bar{x} = \{y \in \mathbb{Z} : y\mathcal{R}x\} = \{y \in \mathbb{Z} : y = x + 3k\}$$

$$= \{x + 3k \mid k \in \mathbb{Z}\}.$$

In particular:

$$\bar{0} = \{y \in \mathbb{Z} : y\mathcal{R}0\} = \{3k \mid k \in \mathbb{Z}\} = 3\mathbb{Z}$$

$$\bar{1} = \{y \in \mathbb{Z} : y\mathcal{R}1\} = \{3k + 1 \mid k \in \mathbb{Z}\} = 3\mathbb{Z} + 1$$

$$\bar{2} = \{y \in \mathbb{Z} : y\mathcal{R}2\} = \{3k + 2 \mid k \in \mathbb{Z}\} = 3\mathbb{Z} + 2.$$

B)

For all $m \in \mathbb{Z}$:

$$\begin{cases} \bar{0} = \overline{3m} \\ \bar{1} = \overline{3m+1} \\ \bar{2} = \overline{3m+2} \end{cases} \quad \text{because} \quad \forall m \in \mathbb{Z} : \begin{cases} 0\mathcal{R}(3m) \\ 1\mathcal{R}(3m+1) \\ 2\mathcal{R}(3m+2) \end{cases}$$

Indeed, for all $m \in \mathbb{Z}$:

$$\begin{cases} 0 - (3m) = 3(-m) \\ 1 - (3m + 1) = 3(-m) \\ 2 - (3m + 2) = 3(-m) \end{cases}, \quad -m \in \mathbb{Z}\,.$$

We have:

$$\begin{cases} \bar{0} \cap \bar{1} = \emptyset \\ \bar{1} \cap \bar{2} = \emptyset \\ \bar{0} \cap \bar{2} = \emptyset \end{cases}, \text{because} \begin{cases} 0 \not\mathcal{R} 1 \\ 1 \not\mathcal{R} 2 \\ 0 \not\mathcal{R} 2 \end{cases} . \text{Indeed,} \begin{cases} 0 - 1 = -1 \neq 3k_1 \\ 1 - 2 = -1 \neq 3k_2 \\ 0 - 2 = -2 \neq 3k_3 \end{cases}, \quad k_1, k_2, k_3 \in \mathbb{Z}\,.$$

We know that:

$$\begin{aligned}\mathbb{Z}/R &= \{\bar{x} : x \in \mathbb{Z}\} \\ &= \{\bar{x} : x = 3m\} \cup \{\bar{x} : x = 3m+1\} \cup \{\bar{x} : x = 3x+2\}. \\ &= \{\bar{0}, \bar{1}, \bar{2}\}.\end{aligned}$$

Ex4 $\forall\,(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2 : x_1 \leqslant x_2$ and $y_1 \leqslant y_2$.

(i) Reflexivity: We know that

$$\forall (x,y) \in \mathbb{R}^2 : \begin{cases} x \leqslant x \\ y \leqslant y \end{cases} \Rightarrow (x,y)R(x,y) \Rightarrow R \text{ is reflexive.}$$

(ii) Anti-symmetry: Suppose $(x_1, y_1)\,R\,(x_2, y_2)$ and $(x_1, y_2)\,R\,(x_1, y_1)$

$$\Rightarrow \begin{cases} x_1 \leqslant x_2 \wedge y_1 \leqslant y_2 \\ \wedge \\ x_2 \leqslant x_1 \wedge y_2 \leqslant y_1 \end{cases} \Rightarrow \begin{cases} x_1 = x_2 \\ \wedge \\ y_1 = y_2 \end{cases} \Rightarrow (x_1, y_1) = (x_2, y_2)\ .\ \text{Thus, } R \text{ is anti-symmetric.}$$

(iii) Transitivity: Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{R}^2$

$$\begin{cases} (x_1, y_1)\,R\,(x_2, y_2) \\ \wedge \\ (x_2, y_2)\,R\,(x_3, y_3) \end{cases} \Rightarrow \begin{cases} x_1 \leqslant x_2 \wedge y_1 \leqslant y_2 \\ \wedge \\ x_2 \leqslant x_3 \wedge y_2 \leqslant y_3 \end{cases} \Rightarrow \begin{cases} x_1 \leqslant x_3 \\ \wedge \\ y_1 \leqslant y_3 \end{cases} \Rightarrow (x_1, y_1)\,R\,(x_3, y_3)$$

Therefore, R is transitive. Hence, R is a partial order relation on $\mathbb{R}^2$.

(2) $(2,4)$ and $(3,1)$ are not comparable because $(1,4)$ and $(3,1)$ do not satisfy the relation. In fact,

$$\begin{cases} 2 \leqslant 3 \\ \wedge \\ 4 \not\leqslant 1 \end{cases} \text{and} \begin{cases} 3 \not\leqslant 2 \\ \wedge \\ 1 \leqslant 2 \end{cases} \Rightarrow \begin{cases} (2,4)\not R(3,1) \\ \wedge \\ (3,1)\not R(2,4) \end{cases}$$

(3) The order is partial because $\exists a = (2,4)$ and $b = (3,1)$ where $a \not R b$ and $b \not R a$.

(4) $t = (x, y) \in \mathbb{R}^2$ is an upper bound of A if $\forall a \in A : aRt$.

$$\Rightarrow \begin{cases} (1,2)R(x,y) \\ \wedge \\ (3,1)R(x,y) \end{cases} \Rightarrow \begin{cases} 1 \leqslant x \wedge 2 \leqslant y. \\ \wedge \\ 3 \leqslant x \wedge 1 \leqslant y. \end{cases} \Rightarrow \begin{cases} x \geqslant 3 \\ \wedge \\ y \geqslant 2 \end{cases}$$

$$\Rightarrow \underset{\mathbb{R}^2}{\mathrm{Maj}}(A) = \{(x, y) : x \geqslant 3 \wedge y \geqslant 2\}.$$