

Solution of Final Exam in Algebra I M'sila
University
License 1 M1 (2024/2025)

January 14, 2025

Exercise 1 () (7 pts)

Are the evaluations true or false? Justify your answers.

1. Let P and Q be two logical propositions, we have:

$$(P \Rightarrow Q) \Leftrightarrow (Q \vee \bar{P})$$

Answer: *True*. This is a known logical equivalence. It can be shown using truth tables or logical identities.

2. Let E be a subset of $\mathbb{R}$ such that $E = \{x \in \mathbb{R} : x^2 - 1 = 0\}$. Then $P(E) = \emptyset$.

Answer: *False*. The set E contains the solutions to $x^2 - 1 = 0$, which are $x = 1$ and $x = -1$. Thus, $E = \{-1, 1\}$, and $P(E)$ is not empty.

3. Let E and F be two sets and $f : E \rightarrow F$ be a function. Then $f(E) = F$.

Answer: *False*. The image $f(E)$ is the set of outputs of f for inputs in E and is not necessarily equal to F .

4. $(3\mathbb{Z}, +)$ is a subgroup of $(\mathbb{Z}, +)$.

Answer: *True*. $3\mathbb{Z}$ is closed under addition, contains the identity element (0) , and every element has an inverse within the set.

or: $\forall x, y \in 3\mathbb{Z}$, then $x - y = 3k - 3k' \in 3\mathbb{Z}$

Exercise 2 () (6° Pts)

We define the relation on $\mathbb{R}^*$ as follows: $\forall x, y \in \mathbb{R}^* : xRy \iff x - y = \frac{1}{x} - \frac{1}{y}$.

1. Show that R is an equivalence relation.

- **Reflexive:** For any $x \in \mathbb{R}^*$, xRx because $x - x = 0 = \frac{1}{x} - \frac{1}{x}$.
- **Symmetric:** If xRy , then $x - y = \frac{1}{x} - \frac{1}{y}$. Rearranging gives $y - x = \frac{1}{y} - \frac{1}{x}$, thus yRx .
- **Transitive:** If xRy and yRz , then $x - y = \frac{1}{x} - \frac{1}{y}$ and $y - z = \frac{1}{y} - \frac{1}{z}$. Adding these gives $x - z = \frac{1}{x} - \frac{1}{z}$, thus xRz .

Therefore, R is an equivalence relation.

2. Determine the equivalence class of 2.
- To find the equivalence class of 2, we solve $2 - y = \frac{1}{2} - \frac{1}{y}$. Rearranging leads to $2y - y^2 + 1 = 0$, which can be solved using the quadratic formula. $\Delta = (-1)^2 - 4(2)(1) = -7 < 0$

Exercise 3 () (7 Pts)

Let $f : \mathbb{Z} \rightarrow \mathbb{Q}$ be a function defined by $f(x) = \frac{x}{2}$.

1. Determine $f(\mathbb{Z})$ and $f^{-1}(B)$ where $B = \{\frac{5}{3}\}$.
 - **Finding $f(\mathbb{Z})$:** The image is $f(\mathbb{Z}) = \{\frac{n}{2} : n \in \mathbb{Z}\}$.
 - **Finding $f^{-1}(B)$:** We need $f(x) = \frac{5}{3}$. Solving gives $x = \frac{10}{3}$, which is not an integer, so $f^{-1}(B) = \emptyset$.
2. Is f injective, surjective?
 - **Injective:** Yes, because if $f(a) = f(b)$, then $\frac{a}{2} = \frac{b}{2}$ implies $a = b$.
 - **Surjective:** No, as $f(\mathbb{Z})$ does not cover all rational numbers.

Good luck!