Finite Probability Spaces

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October 13$^{\text{th}}$, 2025

Random Experiment and Sample Space

Definition

An experiment whose outcome cannot be predicted in advance and may vary under the same conditions.

Definition

The set of all possible outcomes of a random experiment. Denoted as Ω.

Example

- Lifetime of a computer: $\Omega = [0, +\infty)$
- Number of calls in a time interval: $\Omega = \{0, 1, 2, \ldots\}$
- Choosing 2 items without replacement from 3:

$$\Omega = \{(obj1, obj2), (obj1, obj3), (obj2, obj3)\}$$

Events

Definition

An **event** is any subset of the sample space Ω. It occurs if the outcome $\omega \in A$.

Example

- $A_1 = [0, 800]$: lifetime $\leq$ 800 hours
- $A_2 = \{11, 12, 13, 14\}$: between 11 and 14 calls

Elementary and Special Events

Elementary Event: A singleton $\{\omega\}$ from Ω
Special Cases:

- $\emptyset$: Impossible event
- Ω: Certain event

Set Operations on Events

For events A, $B \subset \Omega$:

- Complement: $\overline{A} = \{\omega \in \Omega : \omega \notin A\}$
- Union: $A \cup B$
- Intersection: $A \cap B$
- Incompatible: $A \cap B = \emptyset$
- Implication: If $A \subset B$, then $A \Rightarrow B$

Probability Measure

A probability measure $P : \mathcal{P}(\Omega) \to [0, 1]$ satisfies:

- $P(\Omega) = 1$
- For disjoint A_k:

$$P\left(\bigcup_k A_k\right) = \sum_k P(A_k)$$

The triple $(\Omega, \mathcal{P}(\Omega), P)$ is a **probability space**.

Probability Properties

- $P(\emptyset) = 0$
- $P(\overline{A}) = 1 - P(A)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- If $A \subset B$, then $P(A) \leq P(B)$

Uniform Probability Spaces

If $|\Omega| = n$, and all outcomes equally likely:

$$P(\omega) = \frac{1}{n}, \quad P(A) = \frac{|A|}{n}$$

$$P(A) = \frac{\text{Favorable Cases}}{\text{Total Cases}}$$

Example

From 12 components (4 defective), choose 2:

- Total combinations: $C_{12}^2 = 66$
- Defective: $C_4^2 = 6 \Rightarrow P(A) = \frac{6}{66} = 0.09$
- Non-defective: $C_8^2 = 28 \Rightarrow P(B) = \frac{28}{66} = 0.42$
- At least one defective: $P(C) = 1 - P(B) = 0.58$

Conditional Probability

Definition

Let A and B be two events in a probability space $(\Omega, \mathcal{P}, P)$ such that $P(B) > 0$. The probability of event A occurring given that event B has occurred is called the conditional probability of A given B. It is denoted by $(P(A \mid B)$ and is defined by:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Example

We consider the experiment of rolling a standard six-faced die numbered from 1 to 6, where the faces with even numbers are colored green. The die is rolled, and from a distance, it is observed that the top face is green. What is the probability that the result is a four? The sample space of this experiment is given by:
$\Omega = \{1, 2, 3, 4, 5, 6\}$, Let B be the event 'a green face is obtained' and A the event 'a four is obtained'. We then have:
$B = \{2, 4, 6\}, A = \{4\}$
Then,

$$P(B) = \frac{1}{2}, \quad P(A \cap B) = \frac{1}{6},$$

and so:

$$P(A|B) = \frac{1}{3}$$

Example

Among the third-year licenc's students in SI, 25% failed the exam in **Prob.**, 15% failed the exam in **IHM**, and 10% of the students failed neither IHM or Prob. A student is chosen at random.

- If the student failed in IHM, what is the probability that they also failed in Prob?
- If the student failed in Prob, what is the probability that they also failed in IHM?
- What is the probability that the student failed at least one of the two exams?

Let A be the event "the student failed in Prob." and (B) be the event "the student failed in IHM." Then, we have:

- $P(A) = 0.25$,
- $P(B) = 0.15$,
- $P(A \cap B) = 0.10$.

Example

We want to find:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.10}{0.15} = 0.66$$

We also want to find:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{0.10}{0.25} = 0.4$$

Finally, the probability that the student failed at least one of the two exams is:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.25 + 0.15 - 0.10 = 0.30$$

Independence

Definition

Let A and B be two events in a probability space $(\Omega, \mathcal{P}(\Omega), P)$. We say that the events A and B are **independent** if and only if:

$$P(A \cap B) = P(A) \cdot P(B)$$

Example

In the previous example (rolling a die):

- The event $A = 1, 2, 3$ corresponds to the event that the number obtained is less than or equal to 3.
- The event $B = 1, 3, 5$ corresponds to the event that the number obtained is odd.

We have $A \cap B = \{1, 3\}$, $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{2}$, and

$$P(A \cap B) = \frac{2}{6} \neq P(A) \cdot P(B).$$

In this case, the events A and B are not independent.

Property

the events A and B are ***independent*** *we have:* $P(A|B) = P(A)$
In other words, the occurrence of event B has ***no influence*** *on the occurrence of event A.*

Total Probability Theorem

Theorem

*Let $(A_k)_{k \in K}$ be a **complete system** of mutually exclusive events in the sample space Ω, i.e., a **partition** of the universe. If each event A_k has strictly positive probability $P(A_k) > 0$, then for any event A, we have:*

$$P(A) = \sum_k P(A|A_k) \cdot P(A_k)$$

*This formula is called the **Total Probability Formula***

Bayes' Theorem

Theorem

*Let $(A_k)_{k\in K}$ be a **complete system of mutually exclusive events** in the sample space Ω. If for all $k \in K$, the events A_k have strictly positive probabilities, i.e., $P(A_k) > 0$, and if A is any event such that $P(A) > 0$, then we have:*

$$P(A_k \mid A) = \frac{P(A \mid A_k) \cdot P(A_k)}{\sum_{i\in K} P(A \mid A_i) \cdot P(A_i)}$$

*This formula is called **Bayes' Theorem**, and it plays a **very important role in Bayesian statistics**.*

Example

In a factory, there are three machines A, B, and C that produce mechanical parts of a specific type. Machine A accounts for 25% of the production, machine B for 35%, and machine C for 40%. 5% of the parts produced by machine A are defective. The percentages are 4% and 2% for machines B and C, respectively. A part is randomly selected from a batch made up of parts produced by machines A, B, and C in the given proportions. It is found that the selected part is defective. Calculate the probability that it was produced:

- By machine A
- By machine B
- By machine C

We denote by:

- A: "The part was manufactured by machine A"
- B: "The part was manufactured by machine B"
- C: "The part was manufactured by machine C"
- D: "The part is defective"

Thus, we have: $P(A) = 0.25$, $P(B) = 0.35$, $P(C) = 0.40$ and $P(D|A) = 0.04$, $P(D|B) = 0.02$, $P(D|C) = 0.02$
According to the law of total probability:

$$P(D) = P(D|A)P(A) + P(D|B)P(B) + P(D|C)P(C)$$

$$= (0.04)(0.25) + (0.02)(0.35) + (0.02)(0.40) = 0.025$$

Applying Bayes' Theorem, we find:

* $P(A|D) = \frac{P(D|A)P(A)}{P(D)} = \frac{(0.04)(0.25)}{0.025} = 0.4$ *

$P(B|D) = \frac{(0.02)(0.35)}{0.025} = 0.28$ * $P(C|D) = \frac{(0.02)(0.40)}{0.025} = 0.32$