Basic Set Theory

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10 June 2025

Sets

Definition

A set is a collection of objects, which are called *elements* or *points* of that set.

Definition

Let A and E be two sets. We say that:

- A is a **subset** (or a **part**) of E if every element of A is also an element of E. This is denoted by $A \subset E$, which is read as “A is included in E”.
- The set of all subsets of E is denoted by $\mathcal{P}(E)$, called the **power set** of E.

That is:

$$A \in \mathcal{P}(E) \Rightarrow A \subset E \tag{1}$$

Sets

Remark

A set is generally denoted by an uppercase letter, and its elements are denoted by lowercase letters.

Subset or Part of a Set

Definition

The set of subsets of a set.

Remark

- *$E \in \mathcal{P}(E)$, that is, the set E is an element of the power set $\mathcal{P}(E)$.*
- *$\varnothing \in \mathcal{P}(E)$, i.e., the empty set is also an element of $\mathcal{P}(E)$.*
- *If $x \in E$, then the singleton $\{x\} \in \mathcal{P}(E)$.*

Example

If $E = \{x_1, x_2, \ldots, x_n\}$, then

$$\mathrm{Card}(E) = n \quad \text{and} \quad \mathrm{Card}(\mathcal{P}(E)) = 2^n.$$

Subset or Part of a Set

Example

Let $E = \{x_1, x_2, x_3\}$. Then

$$\mathcal{P}(E) = \{\varnothing, \{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}, \{x_1, x_2, x_3\}\}$$

and

$$\mathrm{Card}(\mathcal{P}(E)) = 2^3 = 8.$$

Operations on Sets

Let A and B be two subsets of a set E. Then we can define:

- $\overline{A}$: the complement of A in E.
- $A \cup B$: the union of A and B,

$$A \cup B = \{x \in E \mid x \in A \vee x \in B\}.$$

- $A \cap B$: the intersection of A and B,

$$A \cap B = \{x \in E \mid x \in A \wedge x \in B\}.$$

- $A \setminus B$: the difference of A and B,

$$A \setminus B = \{x \in E \mid x \in A \wedge x \notin B\} = A \cap \overline{B}.$$

$\sigma-$algebra (Tribe)

Definition

Let E be a set. A subset $\mathcal{A} \subseteq \mathcal{P}(E)$ is called a $\sigma-$**algebra** (or **tribe**) on E if:

- $E \in \mathcal{A}$ (i.e., the whole set belongs to $\mathcal{A}$);
- If $A \in \mathcal{A}$, then $\overline{A} \in \mathcal{A}$ (the complement of A in E);
- If $(A_i)_{i \in I}$ is a countable family (finite or infinite) of elements of $\mathcal{A}$, then
$$\bigcup_{i \in I} A_i \in \mathcal{A}.$$

Definition

Let E be a set and $\mathcal{A}$ a $\sigma-$algebra on E. Then the pair $(E, \mathcal{A})$ is called a **measurable space**.

Example

Let $E = \{x_1, x_2, x_3\}$.

- $\mathcal{P}(E)$ is a σ-algebra of subsets of E.
- $\{\varnothing, E\}$ is a σ-algebra on E.
- $\{\varnothing, \{x_1\}, \{x_2, x_3\}, \{x_1, x_2, x_3\}\}$ is the σ-algebra generated by $\{x_1\}$.

Proposition

Let E be any set.

- $\mathcal{P}(E)$ *is a* σ*-algebra of subsets of* E.
- $\{\varnothing, E\}$ *is a* σ*-algebra on* E. *It is the smallest* $\sigma-$*algebra and is contained in all others.*
- *If* $A \subset E$, *then* $\{\varnothing, E, A, \overline{A}\}$ *is a* σ*-algebra. It is the smallest* σ*-algebra containing* A*; this is the* $\sigma-$*algebra generated by* A.
- *Let* $G \subseteq \mathcal{P}(E)$ *(not necessarily a* σ*-algebra). The* ***σ-algebra generated by*** G *is defined as the intersection of all* σ*-algebras on* E *that contain* G. *It is the smallest* σ*-algebra containing* G.

Definition

A **positive measure** on the measurable space $(E, \mathcal{A})$ is a function

$$m : \mathcal{A} \to [0, +\infty[$$

satisfying the following properties:

- $m(\varnothing) = 0$;
- For every countable (finite or infinite) family $(A_i)_{i\in I}$ of pairwise disjoint sets in $\mathcal{A}$,

$$m\left(\bigcup_{i\in I} A_i\right) = \sum_{i\in I} m(A_i) \tag{2}$$

If $A \in \mathcal{A}$, then the number $m(A)$ (finite or not) is called the **measure of** A. The triple $(E, \mathcal{A}, m)$ is called a **measure space**.

Remark

*Let $\mathcal{G}$ be the family of all open sets. $\mathcal{G}$ is not a σ-algebra on $\mathbb{R}$. The σ-algebra generated by $\mathcal{G}$ is called the **Borel σ-algebra** on $\mathbb{R}$, and is denoted by $\mathcal{B}(\mathbb{R})$.*

Lebesgue–Stieltjes Measure on a σ-algebra

Let $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a measurable space.
Let $f : \mathbb{R} \to \mathbb{R}$ be a non-decreasing, right-continuous function, i.e.,

$$\lim_{\varepsilon \to 0^+} f(x + \varepsilon) = f(x), \quad \forall x \in \mathbb{R}.$$

Proposition

There exists a ***unique measure*** *defined on the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ such that for every bounded semi-open interval $]a,b]$ with $-\infty \leq a < b \leq +\infty$, we have:*

$$m(]a,b]) = f(b) - f(a).$$

This is called the ***Lebesgue–Stieltjes measure*** *associated with f, and it is denoted by m_{LS}.*

Proposition

$$\begin{aligned} m_{LS}(]a,b[) &= f(b^-) - f(a), \\ m_{LS}([a,b[) &= f(b^-) - f(a^-), \\ m_{LS}([a,b]) &= f(b) - f(a^-), \\ m_{LS}(\{a\}) &= f(a) - f(a^-). \end{aligned}$$

If f is discontinuous at a, then $m_{LS}(\{a\}) = 0$.

- Let $f(x) = x$. The associated Lebesgue–Stieltjes measure is the **Lebesgue measure**, denoted m_{Leb}.

$$m_{\text{Leb}}(\{a\}) = 0,$$
$$m_{\text{Leb}}(]a, b]) = m_{\text{Leb}}([a, b[) = m_{\text{Leb}}([a, b]) = m_{\text{Leb}}(]a, b[) = b - a.$$

- Let
$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$
The associated measure is called the **Dirac measure at 0**, denoted m_D. For any Borel set $E \subseteq \mathbb{R}$:
$$m_D(E) = \begin{cases} 1 & \text{if } 0 \in E \\ 0 & \text{otherwise} \end{cases}$$