University of M'sila
Faculty of Mathematics and Computer Science
Department of Mathematics

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Duration: 2 hours

Introduction to Group Theory Exam

Exercise 1 (5 points):

Let G be a finite group of order n.

1. Show that $\forall x \in G : x^n = e$ where e is the identity element of G.
2. Show that if n is prime, then G is cyclic.
3. Let H_1, H_2 be two finite subgroups of G of orders n_1, n_2 respectively. Show that if n_1 and n_2 are coprime, then the intersection $H_1 \cap H_2$ is reduced to $\{e\}$.
4. Let $a \in G$ where $a^2 = e$. Show that the order of a is equal to the order of its inverse a^{-1}.

Exercise 2 (6 points):

Let $GL(2,\mathbb{R}) = \{M \in M(2,\mathbb{R}) : \det(M) \neq 0\}$ and let $SL(2,\mathbb{R}) = \{M \in M(2,\mathbb{R}) : \det(M) = 1\}$.

1. Show that $(SL(2,\mathbb{R}), \times)$ is a subgroup of $(GL(2,\mathbb{R}), \times)$.
2. Let $M = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $N = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ be two elements of $SL(2,\mathbb{R})$.
   Compute $\text{ord}(M)$ and $\text{ord}(N)$.
3. Let f be a map defined by

$$f : (GL(2,\mathbb{R}), \times) \to (\mathbb{R} \setminus \{0\}, \times),\ A \mapsto f(A) = \det(A).$$

$a)$ Show that f is a group homomorphism.
$b)$ Determine $\ker(f)$.

4. Let φ be a map defined by

$$\varphi : (GL(2,\mathbb{R}), \times) \to (GL(2,\mathbb{R}), \times),\ A \mapsto \varphi(A) = PAP^{-1}$$

where $P = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$.
$a)$ Show that φ is a group homomorphism.
$b)$ Show that $\ker(\varphi) = \{I_2\}$ where I_2 is the identity matrix.
$c)$ Is φ injective?

Exercise 3 (5 points):

Let $G = (S_3, \circ)$ with $S_3 = \{e, \tau_1, \tau_2, \tau_3, \sigma_1, \sigma_2\}$ where: $e = (1)$, $\tau_1 = (23)$, $\tau_2 = (13)$, $\tau_3 = (12)$, $\sigma_1 = (123)$, $\sigma_2 = (132)$. Cayley table given

$\circ$	e	τ_1	τ_2	τ_3	σ_1	σ_2
e	e	τ_1	τ_2	τ_3	σ_1	σ_2
τ_1	τ_1	e	σ_1	σ_2	τ_2	τ_3
τ_2	τ_2	σ_2	e	σ_1	τ_3	τ_1
τ_3	τ_3	σ_1	σ_2	e	τ_1	τ_2
σ_1	σ_1	τ_3	τ_1	τ_2	σ_2	e
σ_2	σ_2	τ_2	τ_3	τ_1	e	σ_1

Let $H = \langle \sigma_1 \rangle$.

1. Determine the right cosets modulo H in G ($(G/H)_r$).
2. Determine the left cosets modulo H in G ($(G/H)_\ell$).
3. Deduce that H is normal in G.
4. Compute $(G : H)$, the index of H in G.
5. List the elements of the quotient group S_3/H and find the inverse of the element $\tau_1 H$.
6. Show that $\sigma_1^{2024} \circ \sigma_2^2 = e$.
7. Determine all subgroups of S_3 of order 2.

Exercise 4 (4 points):

Let $G = \langle a \rangle$ be a finite group with $\mid G \mid= 12$.

1. List all generators of G.
2. Find all subgroups of G.
3. Determine the order of a^6.
4. Find all elements of order 4.