**University of M'sila** **2025/2026**
Faculty of Mathematics and Computer Science
Department of Mathematics

# Introduction to Group Theory Exam — Solution

**Exercise 1 (5 points):**

1. Let $G$ be a finite group of order $n$. For any $x \in G$, By Lagrange's theorem, $ord(x) \mid n$. Let$ord(x) = d$, so $d \mid n$. Then $x^n = x^{kd} = \left(x^d\right)^k = e^k = e$ . Hence $x^n = e$. **(1 point)**
2. If $n$ is prime, take any $x \neq e$. Then $|\langle x\rangle|$ divides $n$ and is $> 1$, so $|\langle x\rangle| = n$. Hence $\langle x\rangle = G$, so $G$ is cyclic. **(1.5 points)**
3. $H_1 \cap H_2$ is a subgroup of both $H_1$ and $H_2$. Its order $d$ divides $|H_1| = n_1$ and $|H_2| = n_2$. Since $\gcd(n_1, n_2) = 1$, we get $d = 1$, so $H_1 \cap H_2 = \{e\}$. **(1.5 points)**
4. If $a^2 = e$ then $a = a^{-1}$. Hence $o(a) = o(a^{-1})$ **(1 point)**

**Exercise 2 (6 points):**

1. $SL(2, \mathbb{R})$ $I_2 \neq \phi$ ($I_2$ has determinant 1). If $A, B \in SL(2, \mathbb{R})$, then $\det(AB^{-1}) = \det A \cdot \det B^{-1} = \frac{\det A}{\det B} = \frac{1\cdot}{1} = 1$, so $AB^{-1} \in SL(2, \mathbb{R})$. Hence it is a subgroup. **(1 point)**
2. $M = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ Then $M^2 = I_2$, so $\mathrm{ord}(M) = 2$.
   $N = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. $N^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}. \neq I_2$ for any $k \geq 1$, so $\mathrm{ord}(N) = \infty$.. **(1 point)**

3$a$) $f(AB) = \det(AB) = \det A \cdot \det B = f(A)f(B)$, so $f$ is a homomorphism. **(1 point)**
3$b$) $\ker f = \{A \in GL(2, \mathbb{R}) : \det A = 1\} = SL(2, \mathbb{R})$. **(1 point)**

4$a$) $\varphi(AB) = P(AB)P^{-1} = PAP^{-1}PBP^{-1} = \varphi(A)\varphi(B)$, so homomorphism. **(1 point)**
4$b$) $\ker \varphi = \{A : PAP^{-1} = I_2\} = \{A : A = P^{-1}I_2P = I_2\}$, so $\ker \varphi = \{I_2\}$. **(0.5 point)**
4$c$) Yes, since $\ker \varphi = \{I_2\}$, $\varphi$ is injective. **(0.5 point)**

**Exercise 3 (5 points):**
Here $S_3$ with $\sigma_1 = (123)$, $\sigma_2 = (132)$, $\tau_1 = (23)$, $\tau_2 = (13)$, $\tau_3 = (12)$.
$H = \langle \sigma_1 \rangle = \{e, \sigma_1, \sigma_2\}$**(0.5 point)**

1. Right cosets modulo $H$: $He = H$, $H\tau_1 = \{\tau_1, \sigma_1\tau_1, \sigma_2\tau_1\} = \{\tau_1, \tau_3, \tau_2\}$ (from Cayley table). So $(G/H)_r = \{H, H\tau_1\}$. **(0.5 point)**
2. Left cosets: $eH = H$, $\tau_1 H = \{\tau_1, \tau_1\sigma_1, \tau_1\sigma_2\} = \{\tau_1, \tau_2, \tau_3\}$ (from Cayley table). So $(G/H)_\ell = \{H, \tau_1 H\}$. **(0.5 point)**

3. Since $(G/H)_r = (G/H)_\ell$, so $H$ is normal in $G$. **(0.5 point)**

4. $(G : H) = |G|/|H| = 6/3 = 2$. **(0.5 point)**

5. $S_3/H = \{H, \tau_1 H\}$. $(\tau_1 H)^{-1} = \tau_1^{-1} H = \tau_1 H$ . **(1 point)**

6. $\sigma_1^{2024} = \sigma_1^{3\cdot 674+2} = \sigma_1^2 = \sigma_2$. Then $\sigma_1^{2024} \circ \sigma_2 = \sigma_2 \circ \sigma_2^2 = \sigma_2^3 = e$ (since $ord\,(\sigma_2) = 3$ ). **(0.5 point)**

7. Subgroups of order 2 in $S_3$: generated by transpositions: $\langle \tau_1 \rangle$, $\langle \tau_2 \rangle$, $\langle \tau_3 \rangle$. **(1 point)**

**Exercise 4 (4 points):**
$G = \langle a \rangle$, $|G| = 12$.

1. Generators are $a^k$ with $\gcd(k, 12) = 1$: $k = 1, 5, 7, 11$. Thus $a, a^5, a^7, a^{11}$. **(1 point)**

2. Subgroups correspond to divisors of 12 are $\langle a^{\frac{12}{d}} \rangle$ where $d \mid 12$

   - $d = 1$: $\langle a^{12} \rangle$;
   - $d = 2$: $\langle a^6 \rangle$;
   - $d = 3$: $\langle a^4 \rangle\}$;
   - $d = 4$: $\langle a^3 \rangle$;
   - $d = 6$: $\langle a^2 \rangle$;
   - $d = 12$: $G = \langle a \rangle$.. **(1.5 points)**

3. Order of $a^6$: $\frac{12}{\gcd(6,12)} = \frac{12}{6} = 2$. **(0.5 point)**

4. Elements of order 4 are $a^k$ with $\frac{12}{\gcd(k,12)} = 4 \Rightarrow \gcd(k, 12) = 3$. So $k = 3, 9$. Thus $a^3$ and $a^9$. **(1 point)**