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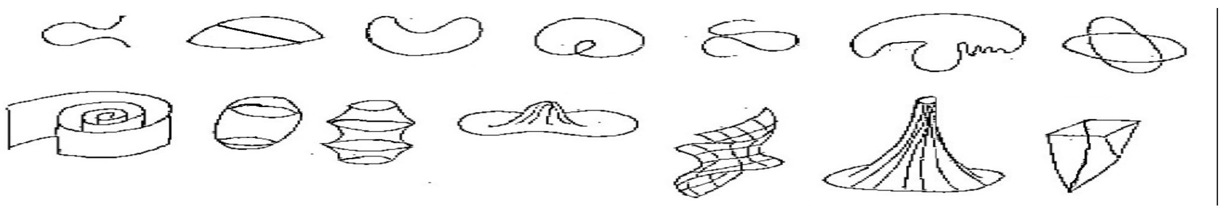
Academic Year: 2025-2026
Module: Differential Geometry
3rd Year Mathematics (S6)

TD Series: Submanifolds, Manifolds, Tangent Space and Bundle

Exercise 01: Determine if the following sets are submanifolds:

1. $M = \{(x, y, z) \in \mathbb{R}^3 : x^3 + y^3 + z^3 - 3xyz = 1\}$ (cubic).
2. $M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \text{ and } x^2 + y^2 - x = 0\}$ (Viviani's window).
3. $M = \{(t, t^2), t \in \mathbb{R}$.
4. $(*)M = \{(x, y) \in \mathbb{R}^2 : x^2 = y^2\}$.
5. $(*)M = \{(x, y) \in \mathbb{R}^2 : x = 0 \text{ or } y = 0\}$.
6. Are the following sets submanifolds?



Exercise 02:

1- Let M be a subset of $\mathbb{R}^2$ defined by:

$$M = \{(x, y) \in \mathbb{R}^2 : y = |x|\}.$$

(a) Show that M is closed in $\mathbb{R}^2$.
(b) Show that M is not a submanifold of $\mathbb{R}^2$.

2- Let S^3 be a subset of $\mathbb{R}^4$ defined by:

$$S^3 = \{(x, y, z, t) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + t^2 = 1\}$$

(a) Show that S^3 is a submanifold of $\mathbb{R}^4$.
(b) What are its dimension and regularity?

Exercise 03:

1. Let the set S^3 in $\mathbb{R}^4$ be defined by:

$$S^3 = \{(x, y, z, t) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + t^2 = 1\}$$

   (a) Is S^3 a submanifold of $\mathbb{R}^4$?
   (b) Determine $T_{(a,b,c,d)}S^3$, the tangent space to S^3 at the point $(a, b, c, d) \in S^3$.

2. Let $M = \{(x, y, z) \in \mathbb{R}^3 : z = x^2 + y^2\}$.

   (a) Calculate $T_{(1,0,1)}M$ by two methods.

3. Let $M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1 \text{ and } 0 < z < 1\}$.

(a) Provide a coordinate system (chart) compatible with the manifold structure.

4. Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$.
   (a) Provide an atlas on S^2 using the North and South stereographic projections.
5. Let $M = \{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$ (a parabola).
   (a) Show that M is a differentiable manifold of dimension 1.
   (b) Determine the tangent space $T_m M$ at the point $m = (1, 1)$ using:
      i. A local parametrization.
      ii. The definition by derivations.

Exercise 04: Let the set M_a be defined as:

$$M_a = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = x \text{ and } x^2 + y^2 + z^2 = a^2,\ a > 0\}.$$

1. Determine a such that M_a is a submanifold.
2. Give a geometric interpretation.
3. Determine the tangent space $T_{(0,0,3)} M_3$.

Exercice 05 :

1. Show that every submanifold is a manifold.
2. Is the converse true?
3. Show that any open set $U \subset \mathbb{R}^n$ is a submanifold of dimension n.
4. Is the converse true?
5. What are the submanifolds of $\mathbb{R}^n$ of dimension 0?

Exercice 06 (*) :

I- Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be of class C^k and $M = \{(x, f(x)), x \in I\}$.

1- Is M a submanifold?

2- If yes, what is its dimension?

II- Show that the image of a submanifold of $\mathbb{R}^n$ by a diffeomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ (φ is of class C^k) is a submanifold of the same dimension.

Exercice 07(*) :

1. Let $GL_n(\mathbb{R}) = \{M \in M_n(\mathbb{R}); M \text{ invertible}\}$.
   (a) $GL_n(\mathbb{R})$ is an open set, and a submanifold.
   (b) If so, what is its dimension?
2. Let $SL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det(A) = 1\}$.
   (a) Show that $SL_n(\mathbb{R})$ is a manifold of dimension $n^2 - 1$.
3. Let $O(n) = \{A \in M_n(\mathbb{R}) : A^t A = I\}$.
   (a) Show that $O(n)$ is a manifold. What is its dimension?

with:

- GL_n: general linear group (the set of invertible matrices).
- SL_n: special linear group (the set of matrices with unit determinant).
- $O(n)$: orthogonal group (the set of orthogonal matrices).

(*) are supplementary exercises.