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Academic Year: 2025-2026
Module: Differential Geometry
3rd Year Mathematics (S6)

TD Series: On Differential Calculus

Exercise 01. Let f be the function defined by: $f(x, y) = x^2y + \sin(xy)$.

1. Compute the first and the second partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial x \partial y}$
2. Verify that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$. What is this equality called?
3. Define $f(x, y) = e^{xy} + x^2 + 2xy$ and $\varphi(t) = f(2t, 3t)$ for $x, y, t \in \mathbb{R}$. Compute $\varphi''(0)$.

Exercise 02. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by: $f(x, y) = (x^2 + y, xy)$.

1. Show that f is differentiable at every point $(x, y) \in \mathbb{R}^2$.
2. Compute the Jacobian matrix of f at a point (x, y).
3. Determine whether f is of class C^1 on $\mathbb{R}^2$.

Exercise 03: Let f be a function from $\mathbb{R}^2$ to $\mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & \text{otherwise} \end{cases}$$

1. Is f continuous at $(0, 0)$?
2. Is f differentiable at $(0, 0)$? (Does f admit partial derivatives at $(0, 0)$?)
3. Show that the following functions are Gâteaux-differentiable:

$$\begin{aligned} f_1 &: \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R}), & A &\mapsto A^2, \\ f_2 &: \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}, & A &\mapsto \operatorname{tr}(A^3). \end{aligned}$$

4. Find the directional derivative of f at point (x_0, y_0) in the direction $v = (a, b)$.
   Application: $f(x, y) = x^2 - y^2$, $(x_0, y_0) = (1, 2)$, and $v = (3, 5)$.

Exercise 04. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad (x, y) \mapsto (e^x \cos y, e^x \sin y)$.

- Show that f is of class C^1 on $\mathbb{R}^2$.
- Compute the Jacobian matrix $Jf_{(x,y)}$ for $(x, y) \in \mathbb{R}^2$.
- Show that f is a local diffeomorphism.
- Is f a global diffeomorphism?

Exercise 05.

1. Consider the function

$$\phi : \mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{R}, \quad (x, y) \mapsto x^y - y^x.$$

- Show that there exist two open intervals I_1 and I_2 containing 1 such that the (non-empty) set
$$E = \{(x,y) \in I_1 \times I_2 : \phi(x,y) = 0\}$$
is the graph of a function $y = \varphi(x)$ of class C^1 on I_1 with values in I_2, satisfying $\varphi(1) = 1$; i.e., $E = \{(x, \varphi(x)), x \in I_1\}$.
- Give the first-order Taylor expansion of φ near 1.

2. (a) Show that there exists an open neighborhood U of the point $(0,0,1)$ in $\mathbb{R}^3$ such that the set
$$\{(x,y,z) \in U : x^2 + y^2 + z^2 - \cos(xyz) = 0\}$$
is the graph of a real-valued function ψ defined on an open set V of $\mathbb{R}^2$.

   (b) Show that ψ is of class C^1 on V and compute its partial derivatives at the point $(0,0)$.

Exercise 06(Additional exercise).

1. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is homogeneous, that is, $f(0) = 0$ and $f(tx) = tf(x)$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$. Prove that if f is differentiable, then $f(x) =< \nabla f(0), x >$.

2. Suppose A is an $m \times n$ matrix. Define $f : \mathbb{R}^n \to \mathbb{R}^m$ by $f(x) = Ax$. Prove that $Df(x) = A$ for all $x \in \mathbb{R}^n$.

3. Suppose $\psi : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable and define $g(s,t) = \psi(s^2 t, s)$. Find $\frac{\partial g}{\partial s}(s,t)$ and $\frac{\partial g}{\partial t}(s,t)$.

4. Suppose $g, h : \mathbb{R}^2 \to \mathbb{R}$ have continuous second derivatives and define
$$u(s,t) = g(s-t) + h(s+t).$$
Prove that $\frac{\partial^2 u}{\partial t^2}(s,t) - \frac{\partial^2 u}{\partial s^2}(s,t) = 0$.