Mohamed Boudiaf University - M'sila
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Duration :2h

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Module: Differential Geometry
3rd Year Mathematics (S6)
Academic Year : 2025-2026

Final test of differential geometry

Exercise 01:(Course Questions)(05pts):

(i) Let M be a submanifold of $\mathbb{R}^n$, the tangent space of M at $m \in M$ noted by .... is ..... of $\mathbb{R}^n$ as the .....

(ii) Let N be a differential manifold of $\mathbb{R}^n$. What is the Atlas (one chart) associated to N in the following cases :

(a) $N = \mathbb{R}^n$.

(b) $N =]0, 1[$.

(iii) Let ω be the 1-form, such that : $\omega = x\,dy - y\,dx + z\,dz$.

(a) Define the domain of definition of ω

(b) Compute $d\omega$.

(c) Is ω closed?

(d) Is ω exact?

(e) Compute $\omega \wedge \eta$ where $\eta = dx \wedge dz$.

Exercise 02 : (7.5pts)

1. Consider the map f, such that :

$$\begin{array}{rcl} f : \mathbb{R}^2 & \rightarrow & \mathbb{R}^2 \\ (x, y) & \mapsto & (x^2 - y^2,\ 2xy) \end{array}$$

(a) Show that f is of class C^1 on $\mathbb{R}^2$.

(b) Write the Jacobian matrix $Jf_{(x,y)}$ for $(x, y) \in \mathbb{R}^2$.

(c) Is f a local diffeomorphism?

(d) Is f a global diffeomorphism?

2. Consider the function $\phi(x, y) = \sin(x + y) + \cos(x - y) - 1$.

(a) Show that there exist two open intervals containing 0 such that the set $E = \{(x, y); \phi(x, y) = 0\}$ is the graph of a function $y = \varphi(x)$ of class C^1, satisfying $\varphi(0) = 0$.

(b) Give the first-order Taylor expansion of φ near 0.

Exercise 03 :(7.5pts)

1. Consider the set $M = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

(a) Show that M is a submanifold $\mathbb{R}^2$.

(b) What are its dimension and regularity?

(c) Determine $T_m M$, the tangent space of M at $m = (1, 0)$.

(d) Exhibit a basis of $T_m M$.

2. Consider the torus $T^2 = S^1 \times S^1 \subset \mathbb{R}^4$, with $S^1 = \{(\cos\theta, \sin\theta)\}$.

(a) Describe T^2 as a set of a function $F : \mathbb{R}^4 \rightarrow \mathbb{R}^2$.

(b) Compute the tangent space $T_m T^2$ for a general point $m = (\cos\theta, \sin\theta, \cos\phi, \sin\phi)$.

(c) Exhibit a basis of $T_m T^2$ when $\theta = 0$.

Scale :(1.5+1+2.5)+((1+1+1+1)+(2.5+1))+((1.5+1+1+1)+(1+1+1)) **Good luck, Mr. B.GAGUI**

Correction of final test of differential geometry

Exercise 01:(Course Questions)(5pts):

(i) Let M be a submanifold of $\mathbb{R}^n$, the tangent space of M at $m \in M$ noted by $T_m M$ is **a vector space** of $\mathbb{R}^n$ as the **same dimension of submanifold** M.

(ii) Let N be a differential manifold. The Atlas associated to N in the following cases :

(a) The simple Atlas associated to N is, $(U, \varphi) = (\mathbb{R}^n, Id_{\mathbb{R}^n})$.

(b) The simple Atlas associated to N is, $(U, \varphi) = (]0,1[, Id_{]0,1[})$.

(iii) Let ω be the 1-form: $\omega = x\,dy - y\,dx + z\,dz$.

(a) The definition domain of ω is $\omega \in \Omega^1_{\mathcal{C}^k(k\geq 1)}(U \subset \mathbb{R}^3, \|\|)$

(b) Recall $d(x\,dy) = dx \wedge dy, \quad d(-y\,dx) = -dy \wedge dx = dx \wedge dy, d(z\,dz) = dz \wedge dz = 0$. Thus $d\omega = (dx \wedge dy) + (dx \wedge dy) + 0 = 2\,dx \wedge dy$.

(c) Since $d\omega = 2dx \wedge dy \neq 0$, ω is **not closed**.

(d) A form must be closed to be exact; hence ω is **not exact**.

(e) Compute $\omega \wedge \eta$ where $\eta = dx \wedge dz$. Compute term by term:

$$(x\,dy) \wedge (dx \wedge dz) = x\,dy \wedge dx \wedge dz = -x\,dx \wedge dy \wedge dz,$$

$$(-y\,dx) \wedge (dx \wedge dz) = -y\,dx \wedge dx \wedge dz = 0, (z\,dz) \wedge (dx \wedge dz) = z\,dz \wedge dx \wedge dz = -z\,dx \wedge dz \wedge dz = 0.$$

Thus $\omega \wedge \eta = -x\,dx \wedge dy \wedge dz$ **is a 3-form**.

Exercise 02 : (7.5pts)

1. Consider the map f, such that :

$$\begin{array}{rcl} f : \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x^2 - y^2,\ 2xy) \end{array}$$

(a) f is polynomial (the components of f), hence f is of class C^1 on $\mathbb{R}^2$.

(b) The Jacobian matrix is:

$$Jf_{(x,y)} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}.$$

(c) The **inverse function theorem applies if** $\det(Jf_{(x,y)}) \neq 0$.

$$\det(Jf_{(x,y)}) = (2x)(2x) - (-2y)(2y) = 4x^2 + 4y^2 = 4(x^2 + y^2).$$

Thus $\det(Jf_{(x,y)}) = 0 \iff (x,y) = (0,0)$. Therefore, for every $(x,y) \neq (0,0)$, f is a **local diffeomorphism in a neighborhood of point** $(0,0)$.

(d) At point $(0,0)$, $\det(Jf_{(x,y)}) = 0$, so f **is not locally invertible**. Indeed, f is **not injective** in any neighborhood of $(0,0)$ because $f(x,y) = f(-x,-y)$. Then f **is not global diffeomorphism**.

2. Consider the function $\phi(x,y) = \sin(x+y) + \cos(x-y) - 1$.

(a) ϕ is a function of class $\mathcal{C}^\infty$.
- $\phi(0,0) = \sin 0 + \cos 0 - 1 = 0 + 1 - 1 = 0$. - $\frac{\partial \phi}{\partial y} = \cos(x+y)\cdot 1 - \sin(x-y)\cdot(-1) = \cos(x+y) + \sin(x-y)$.
At point $(0,0)$: $\frac{\partial \phi}{\partial y}(0,0) = \cos 0 + \sin 0 = 1 \neq 0$.
By the implicit function theorem, there exists a neighborhood I of 0 and a function $y = \varphi(x)$ of class C^1 on I, such that

$$\begin{array}{rcl} E & = & \{(x,y); \phi(x, \varphi(x)) = 0\} \\ E & = & \{(x,y); y = \varphi(x)\} \end{array}$$

$\phi(x, \varphi(x)) = 0 \Rightarrow \phi(0, \varphi(0)) = 0$ then $\varphi(0) = 0$.

(b) Differentiate: $\varphi^{'}(x) = -\frac{\partial\phi/\partial x}{\partial\phi/\partial y}$. Now $\frac{\partial\phi}{\partial x}(x,y) = \cos(x+y)\cdot 1 + (-\sin(x-y))\cdot 1 = \cos(x+y) - \sin(x-y)$.
At $(0,0)$: $\frac{\partial\phi}{\partial x}(0,0) = 1 - 0 = 1$, $\frac{\partial\phi}{\partial y}(0,0) = 1$, so $\varphi'(0) = -1$.
Thus $\varphi(x) = \varphi(0) + \frac{x}{1}\varphi^{'}(0) + O(h)$, then $\varphi(x) \approx -x$ near 0.

Exercise 03 :(7.5pts)

1. Consider the set $M = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

   (a) For every $m \in M$, $m = (x_0, y_0)$, there exists U, $m \in U$.
   Let $f(x,y) = x^2 + y^2 - 1$.
   f is of class C^∞, $M \subset \mathbb{R}^2$. $f^{-1}(0) = U \cap M = \mathbb{R}^2 \cap M = M$.
   Is f a submersion? For every $(x_0, y_0) \in \mathbb{R}^2$,

   $$df_{(x_0,y_0)} = (2x_0, 2y_0) = \nabla(f)$$

   $$\text{rank}(df) = \begin{cases} 1 & x_0 = y_0 \neq 0 \\ 0 & x_0 = y_0 = 0 \end{cases}$$

   $(0,0) \notin S^1$. For every $m \in M$, $\text{rank}(df) = 1 = \dim(\mathbb{R})$, so f is a submersion. Thus, $S^1 = M$ is a submanifold of $\mathbb{R}^2$.

   (b) The dimension is $d = dim(\mathbb{R}^2) -$ Number of equations $= 2 - 1 = 1$ and M is smooth submanifold (i.e., regularity is C^∞).

   (c) Determine $T_m M$, the tangent space of M at $m = (1,0)$.

   $$T_{(1,0)}S^1 = \ker(df_{(1,0)}) = \ker(\nabla f_{(1,0)})$$

   $$df_{(1,0)} : \mathbb{R}^2 \to \mathbb{R}$$

   $$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \mapsto (2\ 0) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = 2h_1$$

   $$\begin{aligned} T_{(1,0)}M &= \left\{(h_1, h_2) \in \mathbb{R}^2 : df_{(1,0)} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = 0\right\} \\ T_{(1,0)}SM &= \{(h_1,h_2) \in \mathbb{R}^2, 2h_1 = 0\} \\ T_{(1,0)}M &= \{(0,h_2) : h_2 \in \mathbb{R}\} \\ T_{(1,0)}M &= \{(0,1)h_2 : h_2 \in \mathbb{R}\}. \end{aligned}$$

   (d) The basis of $T_m M$ when $m = (1,0)$, $T_m M = span\{e_2\} = \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$, where $e_2 = (0,1)$

2. Consider the torus $T^2 = S^1 \times S^1 \subset \mathbb{R}^4$, with $S^1 = \{(\cos\theta, \sin\theta)\}$.

   (a) Define $T^2 = \{(x_1, y_1, x_2, y_2) \in \mathbb{R}^4 : x_1^2 + y_1^2 = 1,\ x_2^2 + y_2^2 = 1\}$ $F = (f_1, f_2)$, $F(x_1, y_1, x_2, y_2) = (x_1^2 + y_1^2 - 1,\ x_2^2 + y_2^2 - 1)$. Then $T^2 = F^{-1}(0,0)$.

   (b)

   $$T_m T^2 = \ker(JF_m) = \ker(JF_m)$$

   . The Jacobian of F is:

   $$JF_X = \begin{pmatrix} 2x_1 & 2y_1 & 0 & 0 \\ 0 & 0 & 2x_2 & 2y_2 \end{pmatrix}.$$

   At point $m = (\cos\theta, \sin\theta, \cos\phi, \sin\phi)$, we have:

   $$JF_m = \begin{pmatrix} 2\cos\theta & 2\sin\theta & 0 & 0 \\ 0 & 0 & 2\cos\phi & 2\sin\phi \end{pmatrix}.$$

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\begin{aligned}
T_mT^2 &= \left\{ (h_1,h_2,h_3,h_4) \in \mathbb{R}^4 : JF_m \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \\
T_mT^2 &= \left\{ (h_1,h_2,h_3,h_4) \in \mathbb{R}^4, 2\cos\theta h_1 + \sin\theta h_2 = 0, 2\cos\phi h_1 + 2\sin\phi h_2 = 0 \right\}.
\end{aligned}
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(c) A basis: $T_mT^2 = span\{v_1, v_2\}$, where $v_1 = (-\sin\theta, \cos\theta, 0, 0)$ and $v_2 = (0, 0, -\sin\phi, \cos\phi)$. When $\theta = 0$,

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T_mT^2 = span\{v_1, v_2\} = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -\sin\phi \\ \cos\phi) \end{pmatrix} \right\rangle
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