Chapter 4

Differential Forms, Exterior Derivative

4.1 Linear and p-Linear Forms

Let E be a vector space of dimension n over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Definition 4.1 *(Linear form) A* ***linear form*** *on a vector space E over a field $\mathbb{K}$ is a map $f : E \to \mathbb{K}$ such that for all vectors $\mathbf{u}, \mathbf{v} \in E$ and all scalars $\alpha, \beta \in \mathbb{K}$,*

$$f(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha f(\mathbf{u}) + \beta f(\mathbf{v}).$$

The set of all linear forms on E forms a vector space called the dual space of E, denoted E^*

Definition 4.2 *Let f be an application such that*

$$f : E \times E \times \cdots \times E \to \mathbb{K}$$

$$(x_1, x_2, \ldots, x_p) \mapsto f(x_1, x_2, \ldots, x_p)$$

f is said to be p-linear if f is linear with respect to each variable $x_i,\ i = 1, 2, ..., p$.

Remarks 4.3

1. *If $p = 1$, f is said to be linear.*
2. *If $p = 2$, f is said to be bilinear.*
3. *The set of p-linear forms is denoted by $L_p(E, \mathbb{K})$.*

Example 4.4

$$f_1 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$$

$$(x, y) \mapsto f_1(x, y) = x_1y_2 - x_2y_1$$

$$f_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$$

$$(x, y) \mapsto f_2(x, y) = x_1y_1 + x_2y_2$$

f_1 and f_2 are bilinear forms on $\mathbb{R}^2 \times \mathbb{R}^2$.

4.2 Alternating p-Linear Forms

Definition 4.5 *Let*

$$f : E \times E \times \cdots \times E = E^p \to \mathbb{K}$$

$$(x_1, x_2, \ldots, x_p) \mapsto f(x_1, x_2, \ldots, x_p)$$

f is said to be an alternating p-linear form if

$$f(x_1, x_2, \ldots, x_i, \ldots, x_j, \ldots, x_p) = -f(x_1, x_2, \ldots, x_j, \ldots, x_i, \ldots, x_p)$$

for all $i, j = 1, p$ and $i \neq j$.

Example 4.6 *In the previous example, f_1 is an alternating form, but f_2 is not.*

Remarks 4.7

1. *The set of alternating p-linear forms is denoted by $A^p(E, \mathbb{K})$ or $A^p(E, \mathbb{K})$.*
2. *$A^p(E, \mathbb{K}) = A^p(E, \mathbb{K})$ is a subspace of $L_p(E, \mathbb{K})$.*

Proposition 4.8 *Let f be such that*

$$f : E \times E \times \cdots \times E = E^p \to \mathbb{K}$$

$$(x_1, x_2, \ldots, x_p) \mapsto f(x_1, x_2, \ldots, x_p)$$

f is alternating if and only if

1. *$f(x_1, x_2, \ldots, x_p) = 0$ whenever there exists a pair (i, j) with $x_i = x_j$ and $i \neq j$.*
2. *For any p-tuple of linearly dependent vectors $x_i = \sum_{j=1}^{p} \lambda_j x_j$.*

Remark 4.9 *If $p > n$, the only alternating p-linear form on E is the zero form $f \equiv 0$.*

Definition 4.10

1. *Let $\{1, 2, \ldots, p\} \subset \mathbb{N}$. A permutation of $\{1, 2, \ldots, p\}$ is any bijective application.*
2. *Let $\sigma \in S(p)$. We say that σ is a transposition if there exist $i, j \in \{1, 2, \ldots, p\}$ such that $\sigma(i) = j$, $\sigma(j) = i$, and $\sigma(k) = k$ for all $k \neq i, k \neq j$.*
3. *We define the application ε by*

$$\varepsilon : S(p) \to \{-1, +1\}$$

$$\sigma \mapsto \varepsilon(\sigma) = \begin{cases} +1, & \text{if the number of permutations is even} \\ -1, & \text{if the number of permutations is odd} \end{cases}$$

The value of ε at σ is called the signature of σ.

Remarks 4.11

1. *A transposition σ is a permutation that swaps two distinct elements and leaves the others unchanged.*

2. $\sigma^2 = I$.

Proposition 4.12 *Let $f \in L_p(E,\mathbb{K})$ and $\sigma \in S(p)$. We define the application (σf) by*

$$(\sigma f) : E \times E \times \cdots \times E = E^p \to \mathbb{K}$$

$$(x_1, x_2, \ldots, x_p) \mapsto f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(p)})$$

which has the following properties:

1. $(\sigma f) = \varepsilon(\sigma) f$, *i.e.,* $f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(p)}) = \varepsilon(\sigma) f(x_1, x_2, \ldots, x_p)$.
2. $\varepsilon(\sigma_1\sigma_2) = \varepsilon(\sigma_1)\varepsilon(\sigma_2)$.
3. $\tau(\sigma f) = (\tau\sigma) f$.

4.3 Exterior Product

Definition 4.13 *Let $f \in A^p(E,\mathbb{K})$, $g \in A^q(E,\mathbb{K})$, and let ϕ be a bilinear application defined by $\phi : F \times G \to H$, where F, G, and H are vector spaces. We define the application $f \wedge_\phi g$ by*

$$f \wedge_\phi g : E^{p+q} \to \mathbb{K}$$

$$(x_1, x_2, \ldots, x_p, x_{p+1}, \ldots, x_{p+q}) \mapsto f \wedge_\phi g$$

where

$$f \wedge_\phi g = \varepsilon(\sigma) \sum_{\sigma \in S(p+q)} \phi\left[f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(p)}), g(x_{\sigma(p+1)}, x_{\sigma(p+2)}, \ldots, x_{\sigma(p+q)})\right]$$

Definition 4.14 *The application $f \wedge_\phi g$ is called the exterior product of f and g with respect to ϕ. We denote by $f \wedge g$ the exterior product of f and g instead of $f \wedge_\phi g$.*

Example 4.15 *Let $f, g \in A^1(E,\mathbb{K})$, $\phi : F \times F \to \mathbb{K}$.*

$$f \wedge_\phi g : E \times E \to \mathbb{K}$$

$$(x_1, x_2) \mapsto f \wedge_\phi g$$

$$f \wedge_\phi g = \sum_{\sigma \in S(2)} \phi\left[f(x_{\sigma(1)}), g(x_{\sigma(2)})\right] = \phi\left[f(x_1), g(x_2)\right] - \phi\left[f(x_2), g(x_1)\right]$$

For

$$\phi : E \times E \to \mathbb{K}$$

$$(x_1, x_2) \mapsto x_1 \cdot x_2$$

we have

$$f \wedge_\phi g = f \wedge g = f(x_1) \cdot g(x_2) - f(x_2) \cdot g(x_1)$$

Definition 4.16 *Let $f_1, f_2, \ldots, f_p$ be p linear forms on E.*

We call the exterior product of p linear forms (f_i), $i \in \{1, 2, \ldots, p\}$ the map

$$\begin{array}{ccc} E \times \cdots \times E & \longrightarrow & \mathbb{K} \\ (x_1, x_2, \ldots, x_p) & \longmapsto & f_1 \wedge f_2 \wedge \cdots \wedge f_p(x_1, x_2, \ldots, x_p) \end{array}$$

such that $f_1 \wedge f_2 \wedge \cdots \wedge f_p(x_1, x_2, \ldots, x_p) = \det[f_i(x_j)]$, $1 \le i, j \le p$.

4.3.1 Properties

1. Let $(e_1, e_2, \dots, e_n)$ be the canonical basis of E and $(e_1^*, e_2^*, \dots, e_n^*)$ its dual basis, then
$$e_i^*(e_j) = \delta_{ij} = \begin{cases} +1, & i = j \\ 0, & i \neq j \end{cases}$$

2. $f_p(x_1, x_2, \dots, x_p) = \det[x_i] f_p(e_1, e_2, \dots, e_n) = \det[x_i]$, $1 \leq i, j \leq n$.

3. Let (e_1, e_2) be a basis of E, $x = x_1 e_1 + x_2 e_2$, $y = y_1 e_1 + y_2 e_2$, then
$$\begin{aligned} f(x,y) &= f(x_1 e_1 + x_2 e_2, y_1 e_1 + y_2 e_2) \\ &= x_1 f(e_1, y_1 e_1 + y_2 e_2) + x_2 f(e_2, y_1 e_1 + y_2 e_2) \\ &= x_1 y_1 f(e_1, e_1) + x_1 y_2 f(e_1, e_2) + x_2 y_1 f(e_2, e_1) + x_2 y_2 f(e_2, e_2) \\ &= x_1 y_2 f(e_1, e_2) + x_2 y_1 f(e_2, e_1) \\ &= x_1 y_2 f(e_1, e_2) - x_2 y_1 f(e_1, e_2) = \det[x_i] f(e_1, e_2) \end{aligned} \tag{4.1}$$
with $f(e_1, e_2) = 1$, $f(e_2, e_1) = -1$, $f(e_1, e_1) = f(e_2, e_2) = 0$.

Proposition 4.17 *Let f, g and h be three p-linear forms, then we have the following properties:*

1. $(f \wedge g) \wedge h = f \wedge (g \wedge h)$ *(the exterior product is associative)*
2. $(f \wedge (g + h)) = (f \wedge g) + (f \wedge h)$ *(the exterior product is distributive)*
3. $f \wedge g = -g \wedge f$
4. *Let $f \in \mathcal{A}^p$ and $g \in \mathcal{A}^q$, then $f \wedge g = (-1)^{pq} g \wedge f$*

4.4 Differential Forms on an Open Set $U \subset \mathbb{R}^n$

Definition 4.18 *We call a differential form of degree p on the open set $U \subset E$ any map ω of the form*
$$\begin{aligned} \omega : U \subset E &\to \mathcal{A}^p(U, \mathbb{K}) \\ x &\mapsto \omega(x) = \sum_{1 \leq i_1, i_2, \dots, i_p \leq n} a_{i_1 i_2 \dots i_p}(x)\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p} \end{aligned}$$
where $a_{i_1 i_2 \dots i_p}$ are continuous functions.

The set of differential forms of degree p on the open set $U \subset E$ of class C^k is denoted by $\Omega^p_{C^k}(U, \mathbb{K})$.

Example 4.19

1. $f : U \subset E \to \mathbb{K}$ *of class C^k, then $f \in \Omega^0_{C^k}(U, \mathbb{K})$.*
2. $f \in C^1$*, then* $df : E \to \mathcal{A}^1(E, \mathbb{K}) \Rightarrow df \in \Omega^1_{C^k}(U, \mathbb{K})$.

Remark 4.20 *Let $\omega \in \Omega^1(U, \mathbb{K})$, then*

$$\begin{aligned} \omega : U \subset E &\to \mathcal{A}^1(U, \mathbb{K}) \\ x &\mapsto \omega(x) \end{aligned}$$

and

$$\begin{aligned} &\omega(x) : E = \mathbb{R}^n \to \mathbb{K} \\ &(\xi_1, \xi_2, \ldots, \xi_n) \mapsto \omega(x)(\xi_1, \xi_2, \ldots, \xi_n) = \omega(x, \xi) \\ &\omega(x, \xi) = \omega\left(x; \sum_{i=1}^{n} \xi_i e_i\right) = \sum_{i=1}^{n} \xi_i \omega(x; e_i) \\ &\qquad = \sum_{i=1}^{n} \omega_i(x) dx_i(\xi) = \left(\sum_{i=1}^{n} \omega_i(x) dx_i\right)(\xi) \end{aligned}$$

then

$$\omega(x) = \sum_{i=1}^{n} \omega_i(x) dx_i$$

is called the canonical form of ω.

4.4.1 Exterior Product of Two Differential Forms

Let ω_1 and ω_2 be two differential forms of degrees p and q (respectively). We define the exterior product $\omega_1 \wedge \omega_2$ by:

$$\begin{aligned} \omega_1 \wedge \omega_2 : \Omega^p(U) \times \Omega^q(U) &\to \Omega^{p+q}(U, \mathbb{K}) \\ (\omega_1, \omega_2) &\mapsto (\omega_1 \wedge \omega_2)(x) = \omega_1(x) \wedge \omega_2(x) \end{aligned}$$

Example 4.21 *Let ω_1 be a 1-form on $\mathbb{R}^3$ defined by $\omega_1(x) = x\,dy - y\,dx - dz$.*

Let ω_2 be a 2-form on $\mathbb{R}^3$ defined by $\omega_2(x) = (x + y)\,dx \wedge dy - z\,dz \wedge dx$. Then $\omega_1 \wedge \omega_2 \in \Omega^3(U \subset \mathbb{R}^3, \mathbb{K})$.

$$\begin{aligned} (\omega_1 \wedge \omega_2)(x) &= (x\,dy - y\,dx - dz) \wedge ((x + y)\,dx \wedge dy - z\,dz \wedge dx) \\ &= x(x + y)\,dy \wedge dx \wedge dy - xz\,dy \wedge dz \wedge dx \\ &\quad - y(x + y)\,dx \wedge dx \wedge dy + yz\,dx \wedge dz \wedge dx \\ &\quad - (x + y)\,dz \wedge dx \wedge dy + z\,dz \wedge dz \wedge dx \\ &= -xz\,dy \wedge dz \wedge dx - (x + y)\,dz \wedge dx \wedge dy \\ &= -xz\,dx \wedge dy \wedge dz + (x + y)\,dx \wedge dy \wedge dz \\ &= (x + y - xz)\,dx \wedge dy \wedge dz \end{aligned}$$

4.5 Exterior Derivative

Definition 4.22 *The map α defined by*

$$\alpha(x)(\xi_1, \xi_2, \ldots, \xi_p) = \sum_{i=1}^{p} (-1)^i d\omega(x, \xi_i)(\xi_1, \xi_2, \ldots, \widehat{\xi_i}, \ldots, \xi_p)$$

where the vector $(\xi_1, \xi_2, \ldots, \widehat{\xi_i}, \ldots, \xi_p)$ is the vector $(\xi_1, \xi_2, \ldots, \xi_p)$ with the component ξ_i omitted. We call α the exterior derivative of ω and denote it by $d\omega$.

Then $d\omega \in \Omega^{p+1}_{C^{k-1}}(U,\mathbb{K})$, *where*

$$d\omega(x;\xi_1,\xi_2,\ldots,\xi_{p+1}) = \sum_{i=1}^{p+1}(-1)^i d\omega(x,\xi_i)(\xi_1,\ldots,\widehat{\xi_i},\ldots,\xi_{p+1})$$

Definition 4.23 *Let* $\omega \in \Omega^p_{C^k}(U)$ *such that* $\omega = \sum_{1\leq i_1<\cdots<i_p\leq n} a_{i_1\ldots i_p} dx_{i_1}\wedge\cdots\wedge dx_{i_p}$. *We define* $d\omega$ *by*

$$d\omega = \sum_{1\leq i_1<\cdots<i_p\leq n} d(a_{i_1\ldots i_p})\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_p}$$

with

$$d(a_{i_1\ldots i_p}) = \sum_{j=1}^{n}\frac{\partial a_{i_1\ldots i_p}}{\partial x_j}dx_j$$

Definition 4.24 *We also define the differential* d *for any* k *by the following properties:*

1. *For* $f \in \Omega^0_{C^k}(U)$, $df = \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(x)dx_i$.
2. *For* $\omega_1 \in \Omega^p_{C^k}(U)$ *and* $\omega_2 \in \Omega^q_{C^k}(U)$, $d(\omega_1\wedge\omega_2) = d\omega_1\wedge\omega_2 + (-1)^p\omega_1\wedge d\omega_2$.
3. $d^2 = 0$.

Remark 4.25 *We define the map* d *by*

$$\begin{aligned} d : \Omega^p_{C^k}(U) &\to \Omega^{p+1}_{C^{k+1}}(U,\mathbb{K}) \\ \omega &\mapsto d\omega \end{aligned}$$

The map d *is linear, i.e.,*

$$d(\lambda\omega_1 + \mu\omega_2) = \lambda d\omega_1 + \mu d\omega_2$$

Example 4.26 *Let* ω *be a 1-form such that*

$$\omega = P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz \in \Omega^1_{C^1}(U,\mathbb{K})$$

and $P, Q, R \in C^1$, *then*

$$\begin{aligned} d\omega = &\left(\frac{\partial P}{\partial x}dx\wedge dx + \frac{\partial P}{\partial y}dy\wedge dx + \frac{\partial P}{\partial z}dz\wedge dx\right) \\ &+ \left(\frac{\partial Q}{\partial x}dx\wedge dy + \frac{\partial Q}{\partial y}dy\wedge dy + \frac{\partial Q}{\partial z}dz\wedge dy\right) \\ &+ \left(\frac{\partial R}{\partial x}dx\wedge dz + \frac{\partial R}{\partial y}dy\wedge dz + \frac{\partial R}{\partial z}dz\wedge dz\right) \end{aligned}$$

thus

$$d\omega = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)dy\wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)dz\wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\wedge dy$$

Remarks 4.27

1. *The associated vector is called the curl vector*

$$\overrightarrow{\mathrm{rot}}(\vec{v}) = \begin{pmatrix} \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \\ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \end{pmatrix}$$

2. *Let* $\omega = a_1 dy \wedge dz + a_2 dz \wedge dx + a_3 dx \wedge dy \in \Omega^2_{C^1}(U, \mathbb{K})$, *then the divergence vector of* $\vec{A} = (a_1, a_2, a_3)$ *denoted* $\mathrm{div}(\vec{A})$ *is defined by:*

$$\mathrm{div}(\vec{A}) = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z} = \sum_{i=1}^{3} \frac{\partial a_i}{\partial x_i}$$

3. *If* $n = 2$, *the* z*-component* $\frac{\partial a_3}{\partial z}$ *is ignored.*

4.5.1 Closed and Exact Forms

Definition 4.28 *Let* $\omega \in \Omega^p_{C^k}(U)$, $p \geq 1$.

1. *We say that* ω *is closed on* U *if* $d\omega = 0$.

2. *We say that* ω *is exact on* U *if there exists* $\alpha \in \Omega^{p-1}_{C^k}(U)$ *such that* $\omega = d\alpha$. *We then say* α *is a primitive of* ω.

Remarks 4.29

1. $\omega = Pdx + Qdy + Rdz \in \Omega^1_{C^1}(U, \mathbb{K})$ *is closed* $\iff \mathrm{rot}(P, Q, R) = \vec{0}$.

2. *Let* $\omega = a_1 dy \wedge dz + a_2 dz \wedge dx + a_3 dx \wedge dy$ *be closed* $\iff \mathrm{div}(\vec{A}) = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z} = 0$.

3. *For* $\omega = \sum_{i=1}^{n} \omega_i dx_i$, *if* $\omega = d\alpha$ *then* $\forall i \in \{1, \dots, n\} : \omega_i = \frac{\partial \alpha}{\partial x_i}$. *Determining* α *reduces to solving a system of partial differential equations.*

4. *If* $\omega = d\alpha$ *is an exact* C^1 *form, then by Schwarz's theorem, for* $i \neq j$,

$$\frac{\partial \omega_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\partial \alpha}{\partial x_i}\right) = \frac{\partial^2 \alpha}{\partial x_i \partial x_j} = \frac{\partial \omega_j}{\partial x_i}$$

5. $\omega = \sum_{i=1}^{n} \omega_i dx_i$ *is a closed differential form if for* $i \neq j$,

$$\frac{\partial \omega_i}{\partial x_j} = \frac{\partial \omega_j}{\partial x_i}$$

4.6 Starry space and Poincaré's Lemma

Definition 4.30 *Let* $U \subset \mathbb{R}^n$ *be an open set. We say that* U *is starry subset (star-shaped) if there exists a point* $a \in U$ *such that for every point* $m \in U$ *the segment* $[am]$ *is contained in* U.

Examples 4.31

1. $\mathbb{R}^2$ *is a starry space of* $\mathbb{R}^2$.

2. *A convex subset of* $\mathbb{R}^2$ *is starry subset.*

3. *The open set* $\mathbb{R}^2 \setminus \{(0,0)\}$ *is not starry subset.*

Theorem 4.32 (Poincaré's Lemma) *If* U *is a star-shaped open set of* $\mathbb{R}^n$, *then every closed differential form on* U *is exact.*