Chapter 3

Tangent Space and Tangent Bundle

3.1 Tangent Space of a Submanifold

Definition 3.1 *Let M be a submanifold of $\mathbb{R}^n$ of dimension d and class C^k.*
A vector $v \in \mathbb{R}^n$ is a tangent vector at m on M if it is the speed (velocity) vector at m of a curve traced on M and originating at m, i.e., if there exists an interval $I =]-\delta, +\delta[$ containing 0 and a differentiable application

$$\gamma :]-\delta, +\delta[\to \mathbb{R}^n, \ \forall t \in I, \gamma(t) \in M$$

$$\gamma(0) = m, \gamma^{'}(0) = \left(\frac{d}{dt}\gamma(t)\right)(0) = v$$

Examples 3.2

1. $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$

$$\gamma :]-\frac{\pi}{2}, \frac{\pi}{2}[\to \mathbb{R}^2$$

$$t \mapsto (\cos t, \sin t)$$

For every $t \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, $\gamma(t) \in S^1 = M$.

$$\gamma(0) = (1, 0) = m$$

$$\gamma^{'}(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}, \gamma^{'}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = v$$

2. $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$

$$\gamma :]-\frac{\pi}{2}, \frac{\pi}{2}[\to \mathbb{R}^3$$

$$t \mapsto (\cos t, \sin t, 0)$$

$$\gamma(0) = (1, 0, 0), \gamma^{'}(t) = \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} \Rightarrow \gamma^{'}(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = v$$

Remark 3.3 *The set of tangent vectors at m on M is called the tangent space at m on M and is denoted by T_mM.*

Proposition 3.4 *T_mM is a vector space of the same dimension as M.*

3.1.1 Method for Computing $T_m M$

Theorem 3.5 *Let M be a submanifold of $\mathbb{R}^n$ of dimension d of class C^k and $m \in M$.*

1. *If in an open neighborhood U of m, $M \cap U = f^{-1}(0)$, where f is a submersion, then*
$$T_m M = \ker df_m = \ker f'(m).$$
2. *If in an open neighborhood U of m there exists a C^k-diffeomorphism φ from U to a neighborhood V of 0 such that $\varphi(U \cap M) = V \cap (\mathbb{R}^d \times \{0\})$, then*
$$T_m M = (\varphi'(m))^{-1} (\mathbb{R}^d).$$
3. *If (Ω, g) is a parametrization of $U \cap M$, where U is a neighborhood of m, then:*
$$T_m M = g'(0)(\mathbb{R}^d) = \operatorname{Im}(g'(0)).$$

Examples 3.6

1. *Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$, $m = (0, 0, 1)$. Compute $T_{(0,0,1)} S^2$?*

   Solution: *We put*
$$f : \mathbb{R}^3 \to \mathbb{R}, \quad (x, y, z) \mapsto f(x, y, z) = x^2 + y^2 + z^2 - 1$$
$$f^{-1}(0) = U \cap M = M, \quad M = S^2$$
Is f a submersion? $\forall x \in \mathbb{R}$, df_x is surjective.
$$d_{(x,y,z)} f = (2x \quad 2y \quad 2z)$$
$$\operatorname{rang}(d_{(x,y,z)} f) = 1 = \dim \mathbb{R}$$
$\implies$ *df_x is surjective* $\implies$ *f is a submersion. Then:*
$$T_{(0,0,1)} S^2 = \ker(f'(0, 0, 1))$$
$$f' = d_{(0,0,1)} f : \mathbb{R}^3 \to \mathbb{R}$$
$$\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} \mapsto (0 \ 0 \ 2) \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = 2h_3$$
$$T_{(0,0,1)} S^2 = \left\{ (h_1, h_2, h_3) \in \mathbb{R}^3 : f'(0, 0, 1) \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = 0 \right\}$$
$$= \{(h_1, h_2, h_3) \in \mathbb{R}^3, 2h_3 = 0\} = \{(h_1, h_2, 0) : h_1, h_2 \in \mathbb{R}\}$$
$T_{(0,0,1)} S^2$ is the plane $z = 0$ (translated to the point m).

2. *Let $M = \{(x,y,z) \in \mathbb{R}^3; z = \varphi(x,y)\}$, where $\varphi : U \subset \mathbb{R}^2 \to \mathbb{R}$ is of class C^k, U is a neighborhood of $(0,0)$ and $\varphi(0,0) = 0$.*
   ***Question:** Compute $T_m M$.*

   Let $m = (x_0, y_0, \varphi(x_0, y_0))$ and define the parametrization $g : \mathbb{R}^2 \to \mathbb{R}^3$ by $g(x,y) = (x, y, \varphi(x,y))$. Then

$$g'(x_0,y_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial\varphi}{\partial x}(x_0,y_0) & \frac{\partial\varphi}{\partial y}(x_0,y_0) \end{pmatrix}$$

$$\operatorname{Im} g'(x_0,y_0) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial\varphi}{\partial x}(x_0,y_0) & \frac{\partial\varphi}{\partial y}(x_0,y_0) \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} : (h_1,h_2) \in \mathbb{R}^2 \right\}$$

   Therefore

$$\begin{aligned} T_m M &= \left\{ \left(h_1, h_2, \frac{\partial\varphi}{\partial x}(x_0,y_0)h_1 + \frac{\partial\varphi}{\partial y}(x_0,y_0)h_2 \right) : (h_1,h_2) \in \mathbb{R}^2 \right\} \\ &= \left\{ (h_1,h_2,h_3) \in \mathbb{R}^3 : h_3 = \frac{\partial\varphi}{\partial x}(x_0,y_0)h_1 + \frac{\partial\varphi}{\partial y}(x_0,y_0)h_2 \right\} \end{aligned}$$

3.1.2 Differentiable Maps

Definition 3.7 *Let M_1, M_2 are two submanifolds of $\mathbb{R}^n$ and $f : M_1 \to M_2$ a map. We say that f is of class C^k at $m \in M_1$ if there exists a neighborhood U of m and a map $g : U \to g(U)$ of class C^k, such that*

$$g|_{U\cap M_1} = f|_{U\cap M_1}.$$

Remarks 3.8

1. *If f is of class C^k at every point of M_1, we say that f is of class C^k.*

2. *If M_1 is an open subset of $\mathbb{R}^n$, we can take $g = f$ and we recover the usual definition.*

Definition 3.9 ***(Tangent Map)*** *Let M be a submanifold of $\mathbb{R}^n$ and $f : M \to \mathbb{R}^p$ a map of class C^1. The tangent map of f at $m \in M$ on M is the restriction of df_m to $T_m M$, denoted by $T_m f$:*

$$T_a f : T_a M \to \mathbb{R}^p, \quad x \mapsto d_a f(x)$$

Definition 3.10 *Let M, N be two submanifolds of $\mathbb{R}^n$ and $f : M \to N$ a map of class C^1. Then the tangent map $T_m f$*

$$T_m f : T_m M \to T_{f(m)} N$$

is defined by: for $v \in T_m M$, with $v = \gamma'(0)$, we have

$$T_m f(v) = (f \circ \gamma)'(0) \in T_{f(m)} N$$

Example 3.11 *If M is an open subset of $\mathbb{R}^n$, then $T_m f$ is none other than df_m considered as a linear map from $\mathbb{R}^n$ to the subspace $df_m(\mathbb{R}^n)$ of $\mathbb{R}^p$.*

Definition 3.12 ***(Immersion/Embedding)*** *Let M be a submanifold of $\mathbb{R}^n$ and $f : M \to \mathbb{R}^p$ a C^k-diffeomorphism from M onto $f(M)$. We say that f is an embedding if $f(M)$ is a submanifold of $\mathbb{R}^p$.*

Theorem 3.13 ***(Composition)*** *If $f : M_1 \to M_2$ and $g : M_2 \to M_3$ are two maps of class C^k between C^k submanifolds, then $g \circ f : M_1 \to M_3$ is a map of class C^k and*

$$T_m(g \circ f) = T_{f(m)}g \circ T_m f.$$

Definition 3.14

1. *We say that $f : M_1 \to M_2$ is a diffeomorphism from the submanifold M_1 onto the submanifold M_2 if f is bijective and both f and f^{-1} are differentiable.*

2. *We say that f is a local diffeomorphism at $m \in M_1$ if there exists an open neighborhood U_{M_1} of m in M_1 (for the topology induced by $\mathbb{R}^n$) and an open neighborhood V_{M_2} of $f(m)$ in M_2 (for the topology induced by $\mathbb{R}^n$) such that the restriction of f to U_{M_1} is a diffeomorphism from U_{M_1} onto V_{M_2}.*

3.1.3 Tangent Space of a Manifold

Definition 3.15 *(Tangent Vector) Let M be a differentiable manifold and m is a point of M. Let γ_1, γ_2 be two parametrized curves defined on $|-1, 1|$ such that $\gamma_1(0) = \gamma_2(0) = m$. We say that γ_1, γ_2 are tangent at m if, for any chart (U, φ) such that $m \in U$, we have*

$$\frac{d}{dt}(\varphi(\gamma_1(t)))_{t=0} = \frac{d}{dt}(\varphi(\gamma_2(t)))_{t=0}.$$

Remarks 3.16

1. *If this equality holds for one chart, it holds for all.*

2. *We sometimes denote the equivalence class of such curves by $J^1\gamma_1(0) = J^1\gamma_2(0)$. The set of equivalence classes of such curves under this relation is called the tangent space at m of M, denoted by T_xM.*

3. *One drawback of this definition is that the vector space structure of T_mM is not evident. Similarly, it is not entirely clear what a continuous (or C^k) family of vector fields is.*

4. *Note, however, the naturality of this definition: if $f : M \to N$ is a differentiable map between manifolds, we define*

$$\begin{array}{rcl} df_m : T_mM & \longrightarrow & T_{f(m)}N \\ \gamma & \mapsto & [f \circ \gamma] \end{array}$$

Remarks 3.17

1. *The set of tangent vectors at m on M is denoted by T_mM.*

2. *T_mM is a vector space of dimension n.*

Definition 3.18 *Let M and N be two manifolds of dimension n and p respectively, and $f : M \to N$ a map of class C^k. The map $T_m f$ from $T_m M$ to $T_{f(m)}N$ is called the tangent map.*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ \varphi(U) \subset \mathbb{R}^n & \xrightarrow{\psi\circ f\circ\varphi^{-1}} & \psi(V) \subset \mathbb{R}^p \end{array}$$

Theorem 3.19 *Let M and N be two manifolds and $f : M \to N$ a map of class C^k. The map $T_m f$ is linear from $T_m M$ to $T_{f(m)}N$. If P is a third manifold and $g : N \to P$ is a map of class C^k, then*

$$T_m(g \circ f) = T_{f(m)}g \circ T_m f.$$

3.1.4 Differentiable Maps

Definition 3.20 *Let M be a differentiable manifold of dimension n and $f : M \to \mathbb{R}^d$ a map. f is said to be differentiable at $m \in M$ if:*

1. *(U, φ) is a chart on M at m.*

2. *$f \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^n \to \mathbb{R}^d$ is differentiable at $\varphi(m)$ on the open set $\varphi(U)$ of $\mathbb{R}^n$.*

Remark 3.21 *This definition is independent of the choice of chart at m.*

Definition 3.22 *Let M and N be two differentiable manifolds of class C^p and dimension n and m respectively and $f : M \to N$ a continuous map.*
We say that f is of class C^r $(r \leq p)$ at $m \in M$ if, for every chart (U, φ) at m and for every chart (V, ψ) at $f(m)$, the map $\psi \circ f \circ \varphi^{-1}$ is of class C^r at $\varphi(m)$ from the open set $\varphi(U \cap f^{-1}(V))$ to $\psi(V)$, i.e., $\psi^{-1} \circ f \circ \varphi : \varphi(U \cap f^{-1}(V)) \to \psi(V)$ is of class C^r at $\varphi(m)$.

Remarks 3.23

1. *We consider $U \cap f^{-1}(V)$ to ensure the compositions are well-defined. It is important to assume f is continuous to guarantee that $\psi^{-1} \circ f \circ \varphi$ is defined on an open subset of $\mathbb{R}^n$ $(\dim M = n)$. U open in M, $f^{-1}(V)$ open in M $\implies$ $\varphi(U \cap f^{-1}(V))$ is open in $\mathbb{R}^n$.*

2. *This definition is independent of the charts chosen at m and $f(m)$.*

Example 3.24 *Let $f : N \to \mathbb{R}$ be a map from a manifold N to $\mathbb{R}$. f is of class C^r on N if $f \circ \varphi^{-1}$ is of class C^r on $\mathbb{R}$ for every chart (U, φ) on N.*

Remark 3.25 *A map $f : M \to N$ from a manifold M to a manifold N is said to be of class C^k if f is of class C^k at every point of M. We write $f \in C^k(M, N)$.*

Proposition 3.26 *Let M and N be two differentiable manifolds of dimension m and n respectively and of class C^k. The following are equivalent:*

1. *$f \in C^k(M, N)$*

2. *$\forall x \in M$, there exists a chart (U, φ) of M at x and a chart (V, ψ) of N at $f(x)$, such that $f(U) \subset V$ and $\psi^{-1} \circ f \circ \varphi \in C^k(\varphi(U \cap f^{-1}(V)), \psi(V))$.*

Proposition 3.27 *Let M, N and P be three differentiable manifolds of dimension m, n and p respectively. If $f \in C^k(M, N)$ and $g \in C^k(N, P)$, then $g \circ f \in C^k(M, P)$.*

Definition 3.28 *Let M, N be two differentiable manifolds and $f : M \to N$ a map. We say that f is a C^k-diffeomorphism if:*

1. *f is of class C^k (i.e., $f \in C^k(M, N)$).*
2. *f is bijective.*
3. *f^{-1} is of class C^k (i.e., $f^{-1} \in C^k(N, M)$).*

Remark 3.29 *We denote by* $\mathrm{Diff}^k(M, N) = \{$*the set of C^k diffeomorphisms from M to N*$\}$.

3.1.5 Tangent Bundle

Let M be a differentiable manifold. To each point $m \in M$ we can associate its tangent vector space T_mM, and then consider the set-theoretic union TM of all these tangent spaces.

Definition 3.30 *Let M be a manifold of dimension n and class C^k. We denote by*

$$TM = \bigcup_{m \in M} T_mM = \bigcup_{m \in M} \{m\} \times T_mM$$

the set of all tangent vectors at every point of M. We call TM the tangent bundle.

Remarks 3.31

1. *$T^*M = \bigcup_{m \in M} T^*_mM$ is called the cotangent bundle.*
2. *T^*_mM is the algebraic dual of T_mM.*

Theorem 3.32 *Let M be a manifold of dimension n and class C^k. The tangent bundle can be endowed with a structure of a differentiable manifold of class C^{k-1} and dimension $2n$ (here $k \geq 2$ is the regularity of M).*