2.3 Manifolds

In this section, we introduce the notion of a differential manifold, which generalizes the notion of a submanifold.

2.3.1 Chart, Atlas

Let M be a set and n a positive integer.

Definition 2.16

1. *A **chart** on M is a bijection $\varphi : U \to V$ of a subset U of M onto an open set V of $\mathbb{R}^n$. We denote the chart by (U, φ, V) or sometimes by (U, φ).*

2. *A **Atlas** on S is a family of charts (U_i, φ_i, V_i) with $\varphi_i : U_i \to V_i$ a homeomorphism, $i \in I$, such that:*

   (a) *$S = \bigcup_{i\in I} U_i$ (a covering).*

   (b) ***Compatibility**: Let two charts (U_i, φ_i), (U_j, φ_j) such that $U_i \cap U_j \neq \emptyset$. Then the map*
   $$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$
   is a C^k-diffeomorphism.

Definition 2.17 *An atlas $\mathcal{A}$ of dimension n on M is a set $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ of charts of dimension n such that:*

1. *The open sets U_α cover M,*

2. *All charts of $\mathcal{A}$ are pairwise compatible.*

Definition 2.18 *Let M be a separated topological space. A **chart system** of dimension n on M is given by:*

1. *An open covering $(U_i)_{i\in I}$ of M,*

2. *A family of homeomorphisms $\varphi_i : U_i \to W_i$ where W_i are open sets of $\mathbb{R}^n$.*

The homeomorphism $\varphi_j^{-1} \circ \varphi_i$ must be a C^k-diffeomorphism from $\varphi_i(U_i \cap U_j)$ onto $\varphi_j(U_i \cap U_j)$.

Definition 2.19 *Two chart systems $\mathcal{U} = \{(U_i, \varphi_i)\}_{i\in I}$ and $\mathcal{V} = \{(V_j, \psi_j)\}_{j\in J}$ are said to be **equivalent** if their union is still a chart system, i.e., for the union*
$$\mathcal{U} \cup \mathcal{V} = \{(U_i, \varphi_i), (V_j, \psi_j)\}_{i\in I, j\in J}$$
the compatibility condition holds. This is equivalent to requiring that $\forall i, j$: $\psi_j \circ \varphi_i^{-1}$ is a C^k-diffeomorphism from $\varphi_i(U_i \cap V_j)$ onto $\psi_j(U_i \cap V_j)$.

Example 2.20

1. *Every open set $U \subset \mathbb{R}^n$ admits an atlas consisting of a single chart (U, Id), hence it is a manifold.*

2. *$\mathbb{S}^n = \{(x_1, x_2, ..., x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + x_2^2 + ... + x_{n+1}^2 = 1\}$ admits an atlas on $\mathbb{S}^n$ formed by two charts (U_N, φ_N) and (U_S, φ_S).*

   (a) *$\mathbb{S}^n = U_N \cup U_S$ (covering).*

   (b) *$\varphi_N \circ \varphi_S^{-1}$ is a diffeomorphism (and similarly $\varphi_S \circ \varphi_N^{-1}$), with (φ_N, φ_S) being the stereographic projections.*

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2.3.2 Topological manifold, differentiable manifold

Definition 2.21 *(Topological manifold) We say that M is a topological manifold if:*

1. *M is a separated topological space.*
2. *For every $m \in M$, there exists an open neighborhood U of m in M and a homeomorphism $\varphi : U \to W \subset \mathbb{R}^n$ where W is an open set of $\mathbb{R}^n$.*

Definition 2.22

1. *A **differentiable structure** of dimension n on M is an equivalence class of atlases of dimension n on M.*
2. *A **differentiable manifold** of dimension n and class C^k is defined as a separated topological space M together with a C^k-chart system of dimension n.*
3. *A **differentiable manifold structure** of dimension n and class C^k on M is the data of an equivalence class of C^k chart systems of dimension n.*

Example 2.23

1. *Every submanifold M of dimension n and class C^k is a differentiable manifold of $\dim = n$ and class C^k. Let M be a submanifold. M is a separated topological space. We know that: $\forall x \in M$, $\exists U$ a neighborhood of x in $\mathbb{R}^n$ and Ω a neighborhood of 0 in $\mathbb{R}^d$, such that: $g(0) = x$, $g'(0)$ is injective (immersion), $g : \Omega \to U \cap M$ is a homeomorphism. $\{U_x \cap M\}_{x \in M}$ is a covering of M. We take $\varphi_i := g_x^{-1} : U \cap M \to \Omega$, which is a C^k-diffeomorphism.*
2. *The converse is false: a square is not a submanifold, however a square is a differentiable manifold.*
3. *The circle $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is a differentiable manifold because it is a submanifold. For the circle, we have:*
   - *(a) $\exists (U_i)_{i \in I}$ a covering of $\mathbb{S}^1$.*
   - *(b) $\varphi : \Omega \to U \cap M$ is a homeomorphism.*
   - *(c) $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cup U_j) \to \varphi_j(U_i \cup U_j)$ is a C^k-diffeomorphism.*

Proposition 2.24 *Let M be a differentiable manifold of dimension n and class C^p with an atlas $\{(U_i, \varphi_i)\}$ on M. Let N be a differentiable manifold of dimension m and class C^q with an atlas $\{(V_j, \psi_j)\}$ on N. Then $M \times N$ is a differentiable manifold of dimension $(n + m)$ and class $C^{\min(p,q)}$.*

Example 2.25 *$M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, 0 < z < 1\}$, M is the cylinder.*

$$M = \mathbb{S}^1 \times]0, 1[$$

$]0, 1[$ is a manifold with $(]0, 1[, id)$ as an atlas. $\mathbb{S}^1$ is a manifold because it is a submanifold. Thus $M = \mathbb{S}^1 \times]0, 1[$ is a manifold.

$$\dim(M) = \dim(\mathbb{S}^1) + \dim(]0, 1[) = 1 + 1 = 2.$$

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Theorem 2.26 *A differentiable manifold M is a locally compact and locally connected space.*

Remarks 2.27

1. *If in all the definitions we replace $\mathbb{R}^n$ by $\mathbb{C}^n$ and C^k-diffeomorphism by biholomorphic map, i.e., (holomorphic + bijective + the inverse is holomorphic), we obtain the notion of a* ***complex analytic manifold****.*

2. *A complex manifold of dimension 1 is called a* ***Riemann surface****.*

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