Chapter 6

Exam Topics and Model Solutions

6.1 Final Exam (University of M'sila)

Exercise 1

1. Let E be a pre-Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and associated norm $\| \cdot \|$. Show that for all $x, y \in E$:

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

2. Give the relationship between normed spaces and pre-Hilbert spaces.

3. Let the space $\mathbb{R}^n$ be equipped with the norm

$$\forall X = (x_1, x_2, ..., x_n) \in \mathbb{R}^n : \quad \|X\|_\infty = \sup_{i \in \{1,2,...,n\}} |x_i|.$$

Is $\mathbb{R}^n$ a pre-Hilbert space?

Exercise 2

Let $E = \mathbb{R}[x]$ be the space of real-coefficient polynomials equipped with the inner product:

$$\forall P, Q \in E : \quad \langle P, Q \rangle = \int_{-1}^{1} P(x)Q(x)\, dx.$$

1. Consider the polynomials $P_0(X) = \frac{1}{\sqrt{2}}$, $P_1(X) = \sqrt{\frac{3}{2}}X$, $P_2(X) = \frac{\sqrt{5}}{2\sqrt{2}}(3X^2 - 1)$.

   Verify that the family $\{P_0, P_1, P_2\}$ is an orthonormal family.

2. Let $F = \text{Vect}\{P_0, P_1, P_2\}$. For every $f \in E$, we denote by $P_F(f)$ the orthogonal projection of f onto F.

   (a) Show that $P_F(f) = \sum_{i=0}^{2} \langle f, P_i \rangle P_i$.

   (b) Let $g(X) = X^3$. Compute $\langle X^3, P_0 \rangle$, $\langle X^3, P_1 \rangle$, $\langle X^3, P_2 \rangle$ and deduce $P_F(g)$.

   (c) Compute the distance from g to F and deduce the value:

$$\min_{a,b,c \in \mathbb{R}} \int_{-1}^{1} (X^3 - aX^2 - bX - c)^2\, dx.$$

Exercise 3

Let $H = L^2([0,1])$ be the space of functions $f : [0,1] \to \mathbb{R}$ such that $\int_0^1 f(x)^2\, dx < \infty$. We assume that $L^2([0,1])$ is a Hilbert space when equipped with the inner product

$$\forall f, g \in H : \quad \langle f, g\rangle = \int_0^1 f(x)g(x)\, dx.$$

1. Let T be a mapping defined by

$$\begin{array}{rcl} T : H & \to & \mathbb{R} \\ f & \mapsto & T(f) = \int_0^1 f(t)\, dt \end{array}$$

   (a) Show that T is a linear and continuous mapping.

   (b) Find a function $g \in H$ such that $T(f) = \langle f, g\rangle$.

2. Let F be a subspace of H defined by:

$$F = \{f \in H : \quad \int_0^1 f(t)\, dt = 0\}$$

   (a) Show that F is closed.

   (b) Verify that $F = \{g\}^\perp$.

   (c) Determine the projection of the function e^t onto F. $\Big($Remember that: $\{g\}^\perp)^\perp = \mathrm{Vect}(\{g\})\Big)$.

6.2 Make-up Exam (University of M'sila)

Exercise 1 Let H be a Hilbert space and let F be a non-empty subset of H.

1. Let $x \in H$. Write a characterization of the orthogonal projection of x onto F (denoted $P_F(x)$) in the case where F is a closed convex set, and in the case where F is a closed subspace.

2. Let $\overline{B}(0,1)$ be the closed unit ball in H.

   (a) Verify that $\overline{B}(0,1)$ is convex.

   (b) Verify that for all $x \in H$, $x \neq 0$, and for all $y \in H$:

$$\Big\langle x - \frac{1}{\|x\|}x, y - \frac{1}{\|x\|}x \Big\rangle = (\|x\| - 1)\Big(\frac{1}{\|x\|}\langle x, y\rangle - 1\Big).$$

   (c) Show that for all $x \in H$, we have

$$P_F(x) = \begin{cases} x, & \text{if } x \in \overline{B}(0,1); \\ \frac{1}{\|x\|}x, & \text{if } x \notin \overline{B}(0,1). \end{cases}$$

Exercise 2 Let $E = C([0,1],\mathbb{R})$ be the space of continuous functions on $[0,1]$, equipped with the inner product defined by

$$\forall f, g \in E: \quad \langle f, g\rangle = \int_0^1 f(t)g(t)\,dt.$$

Let $f_0, f_1 \in E$ be defined by $f_0(t) = 1$, $f_1(t) = t$. Let $F_1 = \text{Vect}\{f_0, f_1\}$, and $F_0 = \text{Vect}\{f_0\}$.

1. Compute $\|f_0\|$ and $P_{F_0}(f_1)$, the orthogonal projection of f_1 onto F_0.

2. By applying the Gram-Schmidt orthonormalization process to the family $\{f_0, f_1\}$, determine an orthonormal basis of F_2, the space generated by f_0 and f_1.

Exercise 3 Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $a \neq 0_H$ an element of H. Let $\{a\}^\perp$ be the orthogonal complement of the set $\{a\}$.

1. (a) Write the definition of $\{a\}^\perp$.

   (b) Show that $\{a\}^\perp$ is a closed subspace.

2. (a) Let $x \in H$ and let $P_F(x)$ be the orthogonal projection of x onto $F = \{a\}^\perp$. Using the characterization of $P_F(x)$, show that there exists $\lambda \in \mathbb{R}$ such that

   $$x = \lambda a + P_F(x).$$

   (Hint: recall that $\left(\{a\}^\perp\right)^\perp = \text{Vect}\{a\}$).

   (b) Verify that:

   $$|\langle x, a\rangle| = |\lambda|\|a\|^2.$$

   (c) Deduce that

   $$d(x, \{a\}^\perp) = \frac{|\langle x, a\rangle|}{\|a\|}.$$

6.3 Final Exam L3-Maths 18-19 (University of M'sila)

Exercise 1 (05 pts)

Let E be a pre-Hilbert space equipped with an inner product $\langle\cdot,\cdot\rangle$ and an associated norm $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$. Let $u, v \in E$, and set:

$$w = u - \frac{\langle u, v\rangle}{\|v\|^2} v, \qquad (v \neq 0).$$

1. Show that $\|w\|^2 = \|u\|^2 - \dfrac{|\langle u, v\rangle|^2}{\|v\|^2}$.

2. Deduce the Cauchy-Schwarz inequality.

3. Write explicitly the Cauchy-Schwarz inequality in the following cases:

   (a) $E = \mathbb{R}^N$, $\langle X, Y\rangle = \sum_{i=1}^N x_i y_i$; $\quad X = (x_1, x_2, ..., x_N), Y = (y_1, y_2, ..., y_N) \in \mathbb{R}^N$.

(b) $E = C([-1,1],\mathbb{R})$, $\langle f,g\rangle = \int_{-1}^{1} f(t)g(t)\,dt$; $\quad f,g \in C([-1,1],\mathbb{R})$.

Exercise 2 (07 pts)

Let $E = C([0,+\infty[,\mathbb{R})$ be the space of continuous functions on $[0,+\infty[$ equipped with the inner product:

$$\forall f,g \in E: \quad \langle f,g\rangle = \int_0^{+\infty} f(x)g(x)e^{-x}\,dx.$$

Given in E the family $\{f_0, f_1\}$ such that $f_0(x) = 1$, $f_1(x) = x$, we consider the subspaces $F_0 = \text{Vect}\{f_0\}$ and $F_1 = \text{Vect}\{f_0, f_1\}$.

1. Determine $\lambda \in \mathbb{R}$ such that $\langle x - \lambda, 1\rangle = 0$.
2. Compute $\|f_0\|$ and $P_{F_0}(f_1)$, the projection of f_1 onto F_0.
3. Determine an orthonormal basis of F_1.

Exercise 3 (08 pts)

Let $H = L^2([-1,1]) = \{f : [-1,1] \to \mathbb{R}$ measurable such that $\int_{-1}^{1} |f(x)|^2\,dx < \infty\}$ equipped with the inner product:

$$\forall f,g \in H: \ \langle f,g\rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$

Let T be the operator defined on H by:

$$\begin{aligned} T : H &\to \mathbb{R} \\ f &\mapsto T(f) = \int_{-1}^{0} f(x)\,dx - \int_0^1 f(x)\,dx. \end{aligned}$$

1. Show that T is linear and continuous.
2. Let $g : [-1,1] \to \mathbb{R}$ be defined by:

$$g(x) = \begin{cases} 1, & \text{if } -1 \le x < 0 \\ -1, & \text{if } 0 \le x \le 1 \end{cases}$$

   (a) Verify that $g \in H$.

   (b) Show that $\{g\}^{\perp} = \ker T$.

   (c) Let $h(x) = e^{-x}$, compute the projection of h onto $\ker T$.

6.4 Make-up Exam L3-Maths 18-19 (University of M'sila)

Exercise 1

Equip $\mathbb{R}^4$ with its standard Euclidean structure. Consider the following vector subspace:

$$F = \Big\{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 : x_1 + x_2 + x_3 + x_4 = 0, \quad x_1 - x_2 + x_3 - x_4 = 0\Big\}.$$

1. Determine a basis of F and deduce an orthonormal basis $\{v_1, v_2\}$ of F.

2. Determine $F^\perp$, the orthogonal complement of F.

3. Let $X \in \mathbb{R}^4$ and let $P_F(X)$ be the orthogonal projection of X onto F.
   Verify that $P_F(X) = \langle X, v_1\rangle v_1 + \langle X, v_2\rangle v_2$. Deduce $P_F(X)$.

4. Compute the distance from the vector $(1,1,1,1)$ to F.

Exercise 2

Let E be the pre-Hilbert space of continuous functions on $[0,1]$ equipped with the usual inner product

$$\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt.$$

Consider the sequence (f_n) defined by

$$f_n(x) = \begin{cases} n, & \text{if } 0 \le x \le \frac{1}{n^3}; \\ \frac{1}{x^{1/3}}, & \text{if } \frac{1}{n^3} < x \le 1. \end{cases}$$

1. Verify that for all $n \in \mathbb{N}^*$, f_n is continuous (i.e., $(f_n) \subset E$).

2. (a) Show that for all $p, q \in \mathbb{N}^*$, with $p > q$, we have:

   $$\|f_p - f_q\|^2 = \frac{1}{p} + \frac{q^2}{p^3} - 2\frac{q}{p^2}.$$

   (b) Deduce that (f_n) is a Cauchy sequence in E.

3. Study the convergence of (f_n) in E. What can you conclude?

Exercise 3

Let $C([0,1],\mathbb{R})$ be the space of continuous functions on $[0,1]$, and let E be the subspace of $C([0,1],\mathbb{R})$ defined by

$$E = \left\{ f \in C([0,1],\mathbb{R}) : \quad \int_0^1 \frac{f^2(t)}{t}\,dt < \infty \right\}.$$

Equip E with the inner product:

$$\langle f, g\rangle = \int_0^1 \frac{f(t)g(t)}{t}\,dt, \qquad \forall f, g \in E. \tag{6.1}$$

1. Let the family $\{a_0 = t, a_1 = t^2\}$, compute $\langle a_0, a_1\rangle$, and $\|a_0\|$.

2. Applying the Gram-Schmidt orthonormalization process to the family $\{t, t^2\}$, compute explicitly the obtained polynomials.

6.5 Final Exam L3-Maths 19-20 (University of M'sila)

Exercise 1 (05pts)

1. Draw a representative diagram to illustrate the relationship between metric spaces, normed spaces, Banach spaces, pre-Hilbert spaces, and Hilbert spaces.

2. Let H be a pre-Hilbert space equipped with an inner product $\langle\cdot,\cdot\rangle$ and the associated norm $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$.

   (a) State the Cauchy-Schwarz inequality.

   (b) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in H and $f \in H$. Using the Cauchy-Schwarz inequality, show that if $\lim_{n\to+\infty} \|f_n - f\| = 0$, then for all $g \in H$, we have $\lim_{n\to+\infty} \langle f_n, g\rangle = \langle f, g\rangle$.

3. State the characterization of the orthogonal projection: onto a closed subspace and onto a closed convex set.

Exercise 2 (08pts)
We aim to compute:

$$\alpha = \inf_{a,b\in\mathbb{R}} \int_0^1 x^2(\log x - ax - b)^2\, dx.$$

To do this, we introduce the space E of continuous functions on $]0,1]$ such that

$$\int_0^1 \Big(xf(x)\Big)^2\, dx < \infty$$

endowed with the inner product

$$\langle f, g\rangle = \int_0^1 x^2 f(x)g(x)\, dx; \quad \forall f, g \in E.$$

Let F be the subspace generated by $\{e_0, e_1\}$ where $e_0(x) = 1$ and $e_1(x) = x$. Consider the function f defined on $]0,1]$ by $f(x) = \log x$.

1. (a) Determine the norm associated with $\langle\cdot,\cdot\rangle$.

   (b) Verify that $f \in E$ and compute $\langle f, e_0\rangle$, $\langle f, e_1\rangle$, $\langle e_0, e_0\rangle$, $\langle e_1, e_1\rangle$, and $\langle e_0, e_1\rangle$.

2. Compute $P_F(f)$, the projection of f onto F.

3. Deduce the value of α.

Exercise 3 (07pts)
Let $E = C([0,1], \mathbb{R}_+)$ be the space of continuous and positive functions on $[0,1]$, endowed with the inner product

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\, dx; \quad \forall f, g \in E.$$

Note that $(E, \|\cdot\|_2)$ is not complete, where $\|\cdot\|_2 = \sqrt{\langle\cdot,\cdot\rangle}$.
Consider the set F defined by:

$$F = \Big\{ f \in E : \quad f(t) = 0, \forall t \in [0, \frac{1}{2}] \Big\}.$$

Let $T : E \to \mathbb{R}$ be the map defined by:

$$T(f) = \int_0^{\frac{1}{2}} f(t)\, dt.$$

1. Show that T is linear and continuous.
2. Verify that F is a closed subspace.
3. Show that $F^\perp = \{f \in E : f(t) = 0, \forall t \in [\frac{1}{2}, 1]\}$. Do we have $F^{\perp\perp} = F$?
4. Show that the subspaces F and $F^\perp$ are not complementary in E, i.e., $E \neq F \oplus F^\perp$. What can you conclude?

6.6 Make-up Exam L3-Maths 19-20 (University of M'sila)

Exercise 1

Let $E = C^1([0,1], \mathbb{R})$. We consider the following application:

$$\begin{array}{rcl} \langle \cdot, \cdot \rangle : E \times E & \to & \mathbb{R} \\ (f, g) & \to & \langle f, g \rangle = f(0)g(0) + \displaystyle\int_0^1 f'(t)g'(t)\, dt. \end{array}$$

1. Show that the application $\langle \cdot, \cdot \rangle$ defines an inner product on E.
2. Compute $\langle \cos, \sin \rangle$, $\|\cos\|$ and $\|\sin\|$.
3. Write the Cauchy-Schwarz inequality in $(E, \langle \cdot, \cdot \rangle)$.

Exercise 2

Let $E = \mathbb{R}[X]$ be the vector space of polynomials with real coefficients, and F the subspace of E consisting of polynomials of degree ≤ 2. For all P and $Q \in E$, we define:

$$\langle P, Q \rangle = \int_0^{+\infty} P(t)Q(t)e^{-t}\, dt.$$

We denote by P_F the orthogonal projection onto F.

1. State the theorem of orthogonal projection onto a closed subspace.
2. Show that for all $n \in \mathbb{N}$, $\int_0^{+\infty} t^n e^{-t}\, dt = n!$.
3. Compute an orthonormal basis (Q_0, Q_1, Q_2) of F by applying the Gram-Schmidt process to the canonical basis $(1, X, X^2)$ of F.
4. Compute $P_F(X^n)$ and $d(X^n, F)$ for all $n \in \mathbb{N}$.

Exercise 3

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, F a closed vector subspace of H, and $\{e_i\}_{i \in \mathbb{N}}$ a Hilbertian basis of F. Let P_F be the orthogonal projector from H onto F. We assume that for all $x \in F$, we have:

$$\|x\|^2 = \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2.$$

1. Simplify $\|x - \sum_{i=1}^{n} |\langle x, e_i\rangle e_i\|^2$.

2. Deduce that for all $x \in F$, we have:

$$x = \sum_{i=1}^{\infty} \langle x, e_i\rangle e_i$$

3. Show that for all $x \in H$, we have:

$$P_F(x) = \sum_{i=1}^{\infty} \langle x, e_i\rangle e_i.$$

6.7 Model Answer for Final Exam L3-Maths 17-18 (University of M'sila)

Exercise 1

1. For all $x, y \in E$, we have:

$$\begin{aligned} \|x+y\|^2 &= \langle x+y, x+y\rangle = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle \quad (6.2) \\ \|x-y\|^2 &= \langle x-y, x-y\rangle = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle \quad (6.3) \end{aligned}$$

   Adding (6.2) and (6.3), we obtain:

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2). \quad (6.4)$$

2. Every pre-Hilbert space is a normed space, but the converse is not always true.

3. If we take $X = (1, 0, ..., 0)$ and $Y = (0, 1, 0, ..., 0)$, then $\|X\|_\infty = 1$, $\|Y\|_\infty = 1$, $X+Y = (1, 1, 0, ..., 0)$, $X-Y = (1, -1, 0, ..., 0)$ hence: $\|X+Y\|_\infty = 1$ and $\|X-Y\|_\infty = 1$. So:

$$\|X+Y\|^2 + \|X-Y\|^2 = 2 \neq 2(\|X\|^2 + \|Y\|^2) = 4$$

   The norm $\|\cdot\|$ does not satisfy the parallelogram identity (6.4) and therefore $\mathbb{R}^n$ is not a pre-Hilbert space.

Exercise 2

Let $E = \mathbb{R}[X]$ be equipped with the inner product:

$$\langle P, Q\rangle = \int_{-1}^{1} P(x)Q(x)\, dx.$$

1. The family $\{P_0, P_1, P_2\}$ is an orthonormal family, because:
   $\langle P_0, P_1\rangle = \frac{\sqrt{3}}{2}\int_{-1}^{1} X\, dX = \left[\frac{\sqrt{3}}{4}X^2\right]_{-1}^{1} = 0$, similarly:
   $\langle P_0, P_2\rangle = \frac{\sqrt{5}}{4}\int_{-1}^{1}(3X^2-1)\, dX = 0$ and $\langle P_1, P_2\rangle = \frac{\sqrt{15}}{4}\int_{-1}^{1}(3X^2-1)\, dX = 0$.
   Also: $\langle P_0, P_0\rangle = \frac{1}{2}\int_{-1}^{1} dX = 1$, $\langle P_1, P_1\rangle = \frac{3}{2}\int_{-1}^{1} X^2\, dX = 1$ and
   $\langle P_2, P_2\rangle = \frac{5}{8}\int_{-1}^{1}(3X^2-1)^2\, dX = 1$.
   Therefore $\forall i, j \in \{0, 1, 2\} : \langle P_i, P_j\rangle = \begin{cases} 1, & \text{if: } i = j; \\ 0, & \text{if: } i \neq j. \end{cases}$

2. Let $F = \text{Span}\{P_0, P_1, P_2\}$.

   (a) According to **Parseval's identity**, for all $f \in E$, we have:

   $$f = \sum_{i=0}^{2} \langle f, P_i \rangle P_i, \tag{6.5}$$

   Since $f_0 = P_F(f) \in F \subset E$, it also satisfies the relation (6.5) as a special case. We then write:

   $$P_F(f) = \sum_{i=0}^{2} \langle P_F(f), P_i \rangle P_i, \tag{6.6}$$

   But by the defining property of the projection onto a subspace, we have:

   $$\forall f \in E : \langle f - P_F(f), h \rangle = 0 \quad \forall h \in F = \text{Span}\{P_0, P_1, P_2\},$$

   which means that $\langle f - P_F(f), P_i \rangle = 0$, hence:

   $$\forall f \in E : \langle f, P_i \rangle = \langle P_F(f), P_i \rangle \tag{6.7}$$

   Plugging (6.7) into (6.6), we get

   $$P_F(f) = \sum_{i=0}^{2} \langle f, P_i \rangle P_i, \forall f \in E. \tag{6.8}$$

   (b) Let $g(X) = X^4$, then:

   - $\langle X^3, P_0 \rangle = \frac{1}{\sqrt{2}} \int_{-1}^{1} X^3 \, dX = 0.$
   - $\langle X^3, P_1 \rangle = \frac{\sqrt{3}}{\sqrt{2}} \int_{-1}^{1} X^4 \, dX = \frac{2\sqrt{3}}{5\sqrt{2}}.$
   - $\langle X^3, P_2 \rangle = \frac{\sqrt{5}}{2\sqrt{2}} \int_{-1}^{1} (3X^2 - 1) X^3 \, dX = 0.$
   - Plugging into (6.8), we get: $P_F(g) = \frac{3}{5} X.$

   (c)

   $$\begin{aligned} d(g, F) = \|P_F(g) - g\| &= \|\frac{3}{5}X - X^3\| \\ &= \left(\langle \frac{3}{5}X - X^3, \frac{3}{5}X - X^3 \rangle \right)^{\frac{1}{2}} \\ &= \left(\int_{-1}^{1} (\frac{3}{5}X - X^3)^2 \right)^{\frac{1}{2}} \\ &= \left(\frac{8}{175} \right)^{\frac{1}{2}} \end{aligned}$$

   We have:

   $$\begin{aligned} \min_{a,b,c \in \mathbb{R}} \int_{-1}^{1} (X^3 - aX^2 - bX - c)^2 \, dX &= \min_{a,b,c \in \mathbb{R}} \|X^3 - aX^2 - bX - c\|^2 \\ &= \|g - P_F(g)\|^2 = \frac{8}{175}. \end{aligned}$$

Exercise 3

1. (a) T is linear, since for all $f, g \in H$, $\alpha \in \mathbb{R}$ we have:

$$T(\alpha f+g)=\int_0^1(\alpha f(t)+g(t))\;dt=\alpha\int_0^1 f(t)\;dt+\int_0^1 g(t)\;dt=\alpha T(f)+T(g).$$

T is continuous. Indeed, for all $f \in H$, we have:

$$|T(f)|=\left|\int_0^1 f(t)\;dt\right|\le\int_0^1|f(t)|\;dt,$$

Applying the Cauchy-Schwarz inequality we find:

$$|T(f)|\le\int_0^1|f(t)|\;dt\le\Big(\int_0^1 dt\Big)^{\frac{1}{2}}\Big(\int_0^1|f(t)|^2\Big)^{\frac{1}{2}}=\|f\|,$$

Hence: $\exists c=1>0, \forall f\in H:|T(f)|\le c\|f\|$.

(b) Take $g=1\in H$; $(\int_0^1 1^2\;dt=1<\infty)$, it satisfies:

$$T(f)=\int_0^1 f(t)\;dt=\langle f,1\rangle.$$

2. (a) We have

$$F=\{f\in H:\int_0^1 f(t)\;dt=0\}=F=\{f\in H:T(f)=0\}=T^{-1}\big(\{0\}\big),$$

Since $\{0\}$ is closed in $\mathbb{R}$ and T is continuous, then F is closed in H.

(b) $F=\{f\in H:T(f)=0\}=\{f\in H:\langle f,g\rangle=0\}=\{g\}^{\perp}$.

(c) Determination of the projection of e^t onto F ($P_F(e^t)$): by the defining property of $P_F(e^t)$, we have:

$$\langle e^t-P_F(e^t),f\rangle=0\quad\forall f\in F=\{g\}^{\perp}\tag{6.9}$$

Relation (6.9) means: $e^t-P_F(e^t)\in\big(\{g\}^{\perp}\big)^{\perp}$, but $\big(\{g\}^{\perp}\big)^{\perp}=\mathrm{Span}(\{g\})$, hence:

$$\exists\lambda\in\mathbb{R}:e^t-P_F(e^t)=\lambda g,$$

so: $P_F(e^t)=e^t-\lambda g=e^t-\lambda$;

Compute λ: since $P_F(e^t)\in F=\{g\}^{\perp}$, then:

$$\begin{aligned}\langle e^t-\lambda,g\rangle=0\;&\Rightarrow\;\int_0^1 e^t-\lambda\;dt=0\\&\Rightarrow\;\lambda=e-1.\end{aligned}$$

therefore $P_F(e^t)=e^t-e+1$.

6.8 Model Answer for Final Exam L3-Math 18-19 (University of M'sila)

Exercise 1

1. By direct computation, we have:

$$\begin{aligned}\|w\|^2 &= \langle w,w\rangle = \langle u - \frac{\langle u,v\rangle}{\|v\|^2}v, u - \frac{\langle u,v\rangle}{\|v\|^2}v\rangle \\ &= \langle u,u\rangle - \frac{\langle u,v\rangle}{\|v\|^2}\langle u,v\rangle - \frac{\langle u,v\rangle}{\|v\|^2}\langle v,u\rangle + \frac{|\langle u,v\rangle|^2}{\|v\|^4}\langle v,v\rangle \\ &= \|u\|^2 - \frac{|\langle u,v\rangle|^2}{\|v\|^2}.\end{aligned}$$

2. Let $u,v \in E$. If $v \neq 0$; noting that $\|w\|^2$ is positive, then we have:

$$0 \leq \|w\|^2 = \|u\|^2 - \frac{|\langle u,v\rangle|^2}{\|v\|^2} \Leftrightarrow \|u\|^2 - \frac{|\langle u,v\rangle|^2}{\|v\|^2} \geq 0 \Leftrightarrow |\langle u,v\rangle|^2 \leq \|u\|^2\|v\|^2$$

and consequently, $|\langle u,v\rangle| \leq \|u\|\|v\|$.

If $v = 0$, then $|\langle u,v\rangle| = \|u\|\|v\| = 0$.
We conclude that $|\langle u,v\rangle| \leq \|u\|\|v\|; \quad \forall u,v \in E$.

3. (a) $\left|\sum_{i=0}^N x_i y_i\right| \leq \left(\sum_{i=0}^N x_i^2\right)^{\frac{1}{2}}\left(\sum_{i=0}^N y_i^2\right)^{\frac{1}{2}}$.

   (b) $\left|\int_{-1}^1 f(t)g(t)\ dt\right| \leq \left(\int_{-1}^1 |f(t)|^2\ dt\right)^{\frac{1}{2}}\left(\int_{-1}^1 |g(t)|^2\ dt\right)^{\frac{1}{2}}$.

Exercise 2

1.

$$\begin{aligned}\langle x-\lambda, 1\rangle = 0 &\Leftrightarrow \int_0^{+\infty} (x-\lambda)1e^{-x}\ dx = 0 \\ &\Leftrightarrow \int_0^{+\infty} xe^{-x}\ dx - \lambda\int_0^{+\infty} e^{-x}\ dx = 0.\end{aligned}$$

We have $\int_0^{+\infty} e^{-x}\ dx = 1$ and by integration by parts we get $\int_0^{+\infty} xe^{-x}\ dx = 1$, thus: $\langle x-\lambda,1\rangle = 0 \Leftrightarrow 1-\lambda = 0 \Leftrightarrow \lambda = 1$.

2. $\|f_0\|^2 = \int_0^{+\infty} 1^2 e^{-x}\ dx = 1$, hence: $\|f_0\| = 1$.

Computation of $P_{F_0}(f_1)$: from the projection theorem, we have

$$f_1 - P_{F_0}(f_1) \in F_0^{\perp} = (Vect\{f_0\})^{\perp},$$

which means that: $\langle f_1 - P_{F_0}(f_1), 1\rangle = 0$.

On the other hand, since $P_{F_0}(f_1) \in F_0 = Vect\{f_0\}$, then there exists $\lambda \in \mathbb{R}$ such that $P_{F_0}(f_1) = \lambda f_0 = \lambda$.

We write: $\langle f_1 - P_{F_0}(f_1), 1\rangle = 0 \Leftrightarrow \langle x-\lambda,1\rangle = 0$, and using question (1) we get $\lambda = 1$. Hence $P_{F_0}(f_1) = 1$.

3. We apply the Gram-Schmidt orthogonalization process:

$$\begin{cases} h_0 = f_0, \\ h_n = f_n - P_{F_{n-1}}(f_n), \quad \text{n=1}. \end{cases}$$

So we only have one case, for $n = 1$: $h_1 = f_1 - P_{F_0}(f_1)$, and we know $P_{F_0}(f_1) = 1$, hence: $h_1 = f_1 - 1$, i.e., $h_1(x) = x - 1$.

Now we construct an orthonormal family; it suffices to divide by the norm. First we have:

$\|h_0\| = \|f_0\| = 1$,

$\|h_1\|^2 = \int_0^{+\infty}(x-1)^2 e^{-x}\, dx = 1 \Rightarrow \|h_1\| = 1$ (by integration by parts).

We have $\frac{h_0}{\|h_0\|} = h_0$ and $\frac{h_1}{\|h_1\|} = h_1$.

Consequently, $\{1, x-1\}$ is an orthonormal family generating F_1, and hence is an orthonormal basis of F_1.

Exercise 3

1. T is linear (from the linearity of the integral).

   T is continuous; let $f \in H$, using the triangle inequality:

$$\begin{aligned} |T(f)| &= \left|\int_{-1}^{0} f(x)\, dx - \int_0^1 f(x)\, dx\right| \\ &\leq \left|\int_{-1}^{0} f(x)\, dx\right| + \left|\int_0^1 f(x)\, dx\right| \\ &\leq \int_{-1}^{0} \Big|f(x)\Big|\, dx + \int_0^1 \Big|f(x)\Big|\, dx \\ &= \int_{-1}^{1} \Big|f(x)\Big|\, dx. \end{aligned}$$

   Applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned} |T(f)| &\leq \left(\int_{-1}^{1} 1^2\, dt\right)^{\frac{1}{2}} \left(\int_{-1}^{1} |f(t)|^2\, dt\right)^{\frac{1}{2}} \\ &= \sqrt{2}\|f\|_H. \end{aligned}$$

   It suffices to take $c = \sqrt{2}$ which satisfies that for all $f \in H$, we have $|T(f)| \leq c\|f\|_H$.

2. (a) $\int_{-1}^{1} \Big|g(x)\Big|^2 dx = \int_{-1}^{0} 1^2\, dx + \int_{-1}^{1}(-1)^2\, dx = 2 < \infty$, hence $g \in H$.

   (b) By definition: $\{g\}^\perp = \{f \in H : \langle f, g\rangle = 0\}$.

$$\begin{aligned} \langle f, g\rangle &= \int_{-1}^{1} f(x)g(x)\, dx \\ &= \int_{-1}^{0} f(x)1\, dx + \int_0^1 f(x)(-1)\, dx \\ &= \int_{-1}^{0} f(x)\, dx - \int_0^1 f(x)\, dx = T(f), \end{aligned}$$

   thus $\langle f, g\rangle = 0 \Leftrightarrow T(f) = 0$. We write then:

$$\{g\}^\perp = \{f \in H : T(f) = 0\} = \ker T.$$

(c) Since T is linear and continuous, then $\ker T$ is a closed subspace of the Hilbert space H, so the projection theorem guarantees that $P = P_{\ker T}(h)$ exists and is unique and satisfies:

$$h - P \in (\ker T)^{\perp}$$

but $(\ker T)^{\perp} = \left(\{g\}^{\perp}\right)^{\perp} = Vect\{g\}$, so there exists $\alpha \in \mathbb{R}$ such that $h(x) - P(x) = \alpha g(x)$ and hence:

$$P(x) = h(x) - \alpha g(x) \tag{6.10}$$

Computation of α: We know that $P \in \ker T$, hence $T(P) = 0$, and we have:

$$\begin{aligned} T(P) = 0 \quad &\Leftrightarrow \quad \int_{-1}^{0} P(x)\,dx - \int_{0}^{1} P(x)\,dx = 0 \\ &\Leftrightarrow \quad \int_{-1}^{0} h(x)\,dx - \alpha \int_{-1}^{0} g(x)\,dx - \int_{0}^{1} h(x)\,dx + \alpha \int_{0}^{1} g(x)\,dx = 0 \\ &\Leftrightarrow \quad \alpha = \frac{1}{2}(e + e^{-1} - 2) \end{aligned}$$

Substituting into (6.10):

$$P(x) = \begin{cases} e^{-x} - \frac{1}{2}(e + e^{-1} - 2), & \text{if } -1 \leq x < 0; \\ e^{-x} + \frac{1}{2}(e + e^{-1} - 2), & \text{if } 0 \leq x \leq 1. \end{cases}$$

Remark: To compute α, one may also use the fact that $P \in \{g\}^{\perp}$, i.e., $\langle P, g\rangle = 0$, and complete the computation similarly.

6.9 Model Answer for the Remedial Exam L3-Maths 18-19 (University of M'sila)

Exercise 1

1. We have: $F = \text{Vect}\{(1,0,-1,0),(0,1,0,-1)\}$.
   We can construct an orthonormal basis by dividing by the norm, thus $\mathcal{B} = \{v_1 = \frac{1}{\sqrt{2}}(1,0,-1,0), v_2 = \frac{1}{\sqrt{2}}(0,1,0,-1)\}$ is an orthonormal basis of F.

2. By definition, $F^{\perp} = \{Y = (y_1,y_2,y_3,y_4) \in \mathbb{R}^4;\ \langle Y, v_1\rangle = 0 \text{ and } \langle Y, v_2\rangle = 0\}$.

   $$\begin{cases} \langle Y, v_1\rangle = 0 \Rightarrow \frac{1}{\sqrt{2}}y_1 - \frac{1}{\sqrt{2}}y_3 = 0 \Rightarrow y_1 = y_3, \\ \langle Y, v_2\rangle = 0 \Rightarrow \frac{1}{\sqrt{2}}y_2 - \frac{1}{\sqrt{2}}y_4 = 0 \Rightarrow y_2 = y_4, \end{cases}.$$

   Hence, $F^{\perp} = \text{Vect}\{(1,0,1,0),(0,1,0,1)\}$.

3. Let $X \in \mathbb{R}^4$. According to the projection theorem, we have $\langle X - P_F(X), v_1\rangle = 0$ and $\langle X - P_F(X), v_2\rangle = 0$, thus

   $$\langle X, v_1\rangle = \langle P_F(X), v_1\rangle, \qquad \langle X, v_2\rangle = \langle P_F(X), v_2\rangle \tag{6.11}$$

   Moreover, we know that $P_F(X) \in F = \text{Vect}\{v_1, v_2\}$, so there exist $\alpha_1, \alpha_2 \in \mathbb{R}$ such that

   $$P_F(X) = \alpha_1 v_1 + \alpha_2 v_2. \tag{6.12}$$

Substituting in equations (6.11) and (6.12) and using the orthonormality of the basis $\{v_1, v_2\}$, we get

$$\langle X, v_1\rangle = \langle \alpha_1 v_1 + \alpha_2 v_2, v_1\rangle \;\Rightarrow\; \alpha_1 = \langle X, v_1\rangle$$
$$\langle X, v_2\rangle = \langle \alpha_1 v_1 + \alpha_2 v_2, v_2\rangle \;\Rightarrow\; \alpha_2 = \langle X, v_2\rangle$$

Plugging into (6.12) we obtain: $P_F(X) = \langle X, v_1\rangle v_1 + \langle X, v_2\rangle v_2$.

Knowing that $\langle X, v_1\rangle = \frac{1}{\sqrt{2}}(x_1 - x_3)$ and $\langle X, v_2\rangle = \frac{1}{\sqrt{2}}(x_2 - x_4)$, we deduce that:

$$P_F(X_0) = \frac{1}{2}(x_1 - x_3, x_2 - x_4, -x_1 + x_3, -x_2 + x_4). \tag{6.13}$$

4. Let $X_0 = (1,1,1,1)$. We have $d(X_0, F) = \|X_0 - P_F(X_0)\|$, using (6.13) we find $P_F(X_0) = (0,0,0,0)$, hence:
$$d(X_0, F) = \|X_0\| = 2.$$

Exercise 2

1. Obvious.

2. (a) Let $p, q \in \mathbb{N}$ such that $p > q$. To simplify things, we rely on the following table:

x	0	$\frac{1}{p^3}$	$\frac{1}{q^3}$	1
$f_p(x)$	p	$\frac{1}{x^{1/3}}$	$\frac{1}{x^{1/3}}$	
$f_q(x)$	q	q	$\frac{1}{x^{1/3}}$	
$\lvert f_p(x) - f_q(x)\rvert^2$	$\lvert p-q\rvert^2$	$\lvert\frac{1}{x^{1/3}} - q\rvert^2$	0	

Table 6.1: Table to test captions and labels

Hence:

$$\begin{aligned}
\|f_p(x) - f_q(x)\|^2 &= \int_0^1 |f_p(x) - f_q(x)|^2\, dx \\
&= \int_0^{\frac{1}{p^3}} |p-q|^2\, dx + \int_{\frac{1}{p^3}}^{\frac{1}{q^3}} |\frac{1}{x^{1/3}} - q|^2\, dx \\
&= \frac{1}{p} + \frac{q^2}{p^3} - 2\frac{q}{p^2}.
\end{aligned}$$

(b) Let $p, q \in \mathbb{N}$ such that $p > q$, we have

$$\begin{aligned}
\|f_p(x) - f_q(x)\|^2 &= \frac{1}{p} + \frac{q^2}{p^3} - 2\frac{q}{p^2} \\
&\leq \frac{1}{q} + \frac{q^2}{q^3} = \frac{2}{q} \to 0 \quad \text{as } q \to +\infty.
\end{aligned}$$

thus (f_n) is a Cauchy sequence in E.

3. It is easy to see that the pointwise limit of $f_n(x)$ is $+\infty$, hence the sequence (f_n) diverges in E.
We conclude that the space E is not complete (i.e., E is not a Hilbert space).

Exercise 3

1. $\langle a_0, a_1\rangle = \int_0^1 \frac{t\times t^2}{t}\,dt = \frac{1}{3}$, $\|a_0\|^2 = \int_0^1 \frac{t^2}{t}\,dt = \frac{1}{2} \Rightarrow \|a_0\| = \frac{1}{\sqrt{2}}$.

2. We apply the Gram-Schmidt orthogonalization process.

$$\begin{cases} b_0 = a_0, \\ b_n = a_n - P_{F_{n-1}}(a_n), \quad \text{n=1.} \end{cases}$$

where $F_0 = \text{Vect}\{a_0\}$ and $F_1 = \text{Vect}\{a_0, a_1\}$.
We have $b_1 = a_1 - P_{F_0}(a_1) = t^2 - P_{F_0}(t^2)$, since $P_{F_0}(t^2) \in F_0$, then there exists $\alpha \in \mathbb{R}$ such that

$$P_{F_0}(t^2) = \alpha t. \tag{6.14}$$

According to the projection theorem, $\langle t^2 - P_{F_0}(t^2), t\rangle = 0$, hence:

$$\langle t^2 - P_{F_0}(t^2), t\rangle = 0 \Leftrightarrow \langle t^2 - \alpha t, t\rangle = 0 \Leftrightarrow \langle t^2, t\rangle = \alpha\langle t, t\rangle$$

which gives: $\alpha = \frac{\langle t^2, t\rangle}{\|t\|^2} = \frac{2}{3}$. Substituting into (6.14) we get $P_{F_0}(t^2) = \frac{2}{3}t$.

So $b_1 = t^2 - \frac{2}{3}t$.

Now, we normalize the obtained family:

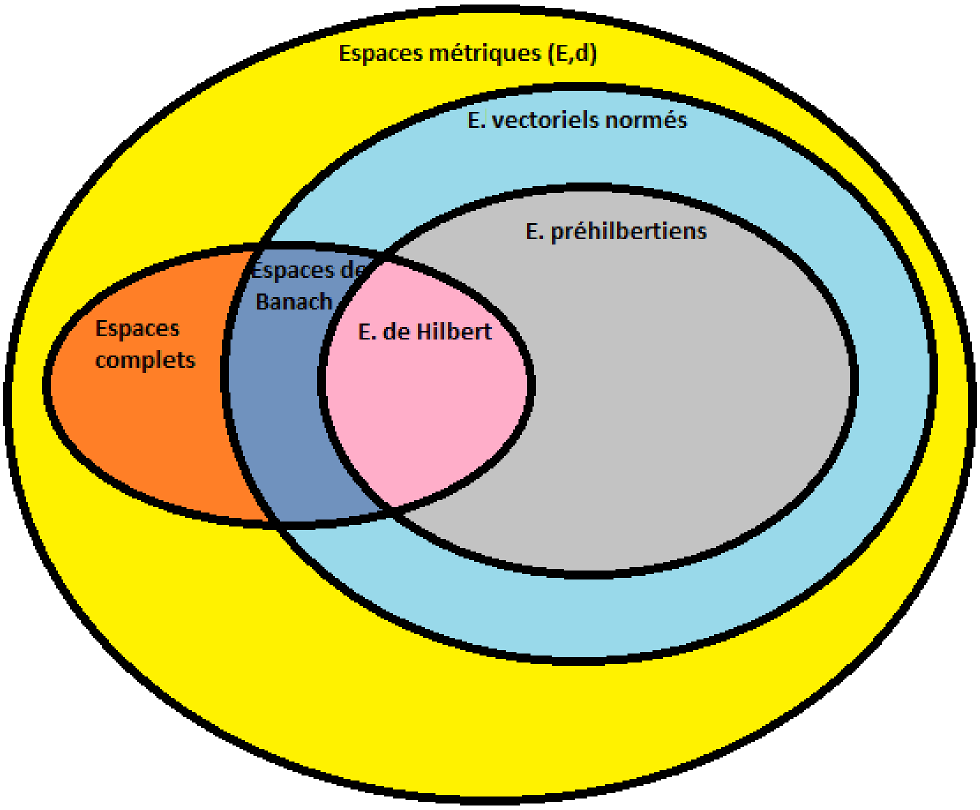
- $b_0' = \frac{b_0}{\|b_0\|} = \frac{a_0}{\|a_0\|} = \sqrt{2}t$.
- $b_1' = \frac{b_1}{\|b_1\|}$; we compute $\|b_1\|$:

$$\|b_1\|^2 = \int_0^1 \frac{(t^2 - \frac{2}{3}t)^2}{t}\,dt = \frac{1}{36} \Rightarrow \|b_1\| = \frac{1}{6}.$$

and hence $b_1' = \frac{1}{6}t^2 - \frac{1}{9}t$.

6.10 Model Answer for Final Exam L3-Maths 19-20

Exercise 1



1. .

2. (a) $\forall x, y \in H : |\langle x, y\rangle| \leq \sqrt{\langle x, x\rangle}\sqrt{\langle y, y\rangle} = \|x\|\|y\|$.

   (b) Suppose that $\lim_{n\to+\infty} \|f_n - f\| = 0 \quad (*)$. By applying the Cauchy-Schwarz inequality, for any $g \in H$, we have:

   $$|\langle f_n, g\rangle - \langle f, g\rangle| = |\langle f_n - f, g\rangle| \leq \|f_n - f\|\|g\|.$$

   Since $g \in H$ and by assumption $(*)$, we deduce that $\lim_{n\to+\infty} |\langle f_n, g\rangle - \langle f, g\rangle| = 0$ and thus $\lim_{n\to+\infty} \langle f_n, g\rangle = \langle f, g\rangle$.

3. Characterization of the orthogonal projection denoted P_F:

   (a) Onto a closed subspace: $x' = P_F(x) \Leftrightarrow \langle x - x', y\rangle = 0$, $\forall y \in F$.

   (b) Onto a closed convex set: $x' = P_F(x) \Leftrightarrow \langle x - x', x - y\rangle \leq 0$, $\forall y \in F$.

Exercise 2

1. (a) $\forall g \in E : \|g\| = \sqrt{\langle g, g\rangle} = \sqrt{\int_0^1 x^2 (g(x))^2 \, dx}$.

   (b)
   - f is continuous on $]0, 1]$, moreover, by integrating by parts twice we obtain

     $$\int_0^1 (x \log x)^2 \, dx = \frac{2}{27} < \infty.$$

     So we deduce that $f \in E$.
     Note: One can use Bertrand's criterion to ensure that $\int_0^1 (x \log x)^2 \, dx < \infty$.
   - By integration by parts: $\langle f, e_0\rangle = \int_0^1 x^2 \log x \, dx = \frac{-1}{9}$.

- Similarly: $\langle f, e_1\rangle = \int_0^1 x^3 \log x \, dx = \frac{-1}{16}$.
- $\langle e_0, e_0\rangle = \int_0^1 x^2 \, dx = \frac{1}{3}$.
- $\langle e_1, e_1\rangle = \int_0^1 x^4 \, dx = \frac{1}{5}$.
- $\langle e_0, e_1\rangle = \int_0^1 x^3 \, dx = \frac{1}{4}$.

2. Calculation of $P_F(f)$: First, we know that $P_F(f) \in F = \text{Span}\{e_0, e_1\}$, so there exist $a_0, b_0 \in \mathbb{R}$ such that
$$P_F(f) = b_0 e_0 + a_0 e_1 = b_0 + a_0 x.$$
On the other hand, from the characterization of $P_F(f)$, we have:
$$\begin{cases} \langle f - P_F(f), e_0\rangle = 0 \\ \langle f - P_F(f), e_1\rangle = 0 \end{cases} \Leftrightarrow \begin{cases} \langle f - b_0e_0 - a_0e_1, e_0\rangle = 0, \\ \langle f - b_0e_0 - a_0e_1, e_1\rangle = 0 \end{cases}$$
From which:
$$\begin{cases} \langle f, e_0\rangle - b_0\langle e_0, e_0\rangle - a_0\langle e_1, e_0\rangle = 0, \\ \langle f, e_1\rangle - b_0\langle e_0, e_1\rangle - a_0\langle e_1, e_1\rangle = 0 \end{cases}$$
Using the results from (2-b), we get:
$$\begin{cases} -\frac{1}{9} - \frac{1}{3}b_0 - \frac{1}{4}a_0 = 0, \\ -\frac{1}{16} - \frac{1}{4}b_0 - \frac{1}{5}a_0 = 0 \end{cases} \Leftrightarrow \begin{cases} 4b_0 + 3a_0 = -\frac{4}{3}, \\ 5b_0 + 4a_0 = -\frac{5}{4} \end{cases} \Leftrightarrow \begin{cases} a_0 = \frac{5}{3}, \\ b_0 = -\frac{19}{12} \end{cases}$$
Consequently: $P_F(f) = \frac{5}{3}x - \frac{19}{12}$.

3. We note that $\alpha = \inf_{p\in F} \|f - p\|^2 = \|f - P_F(f)\|^2$.

By direct calculation and using results from (2-b), we get:
$$\begin{aligned} \|f - P_F(f)\|^2 &= \int_0^1 x^2(\log x - \frac{5}{3}x + \frac{19}{12})^2 \, dx \\ &= \int_0^1 x^2 \log^2 x \, dx + \frac{25}{9}\int_0^1 x^4 \, dx + \frac{361}{144}\int_0^1 x^2 \, dx - \frac{95}{18}\int_0^1 x^3 \, dx \\ &\quad -\frac{10}{3}\int_0^1 x^3 \log x \, dx + \frac{19}{6}\int_0^1 x^2 \log x \, dx \\ &= \frac{1}{432} \end{aligned}$$
Alternative method to avoid calculation: we note that
$$\|f - P_F(f)\|^2 = \langle f - P_F(f), f - P_F(f)\rangle = \|f\|^2 - \langle P_F(f), f\rangle$$
We have $\|f\|^2 = \frac{2}{27}$ and $\langle P_F(f), f\rangle = \frac{5}{3}\langle e_1, f\rangle - \frac{19}{12}\langle e_0, f\rangle$. Hence $\|f - P_F(f)\|^2 = \frac{1}{432}$.

Exercise 3

1. T is linear because: for all $\alpha, \beta \in \mathbb{R}_+$ and for all $f, g \in E$, we have
$$T(\alpha f + \beta g) = \int_0^{\frac{1}{2}} (\alpha f + \beta g)(t)dt = \alpha \int_0^{\frac{1}{2}} f(t)dt + \beta \int_0^{\frac{1}{2}} g(t)dt = \alpha T(f) + \beta T(g).$$

T is continuous because for all $f \in E$, using Cauchy-Schwarz:

$$\begin{aligned} |T(f)| &= \left| \int_0^{\frac{1}{2}} f(t)\, dt \right| \\ &\leq \int_0^1 f(t)\, dt \\ &\leq \int_0^1 \big(f(t)\big)^2\, dt = \|f\|, \end{aligned}$$

thus we can take $c = 1$.

2. F is a subspace: $0 \in F$. For all $\alpha, \beta \in \mathbb{R}_+$ and for all $f, g \in F$, we have $\alpha f + \beta g \in F$ because $(\alpha f + \beta g)(t) = \alpha f(t) + \beta g(t) = 0$, $\forall t \in [0, \frac{1}{2}]$.

   We have $T(f) = 0 \Leftrightarrow f(t) = 0, \forall t \in [0, \frac{1}{2}]$. Thus:

   $$F = \{f \in E : T(f) = 0\} = T^{-1}(\{0\}).$$

   Hence F is closed as the preimage of a closed set under a continuous function.

3. - By definition:

     $$F^{\perp} = \{f \in F : \langle f, g\rangle = 0, \forall g \in F\}.$$

     So we will show that $\{f \in E : \langle f, g\rangle = 0, \forall g \in F\} = \{f \in E : f(t) = 0, \forall t \in [\frac{1}{2}, 1]\} \doteqdot B$.

     Let $f \in B$ then $f(t) = 0, \forall t \in [\frac{1}{2}, 1]$, hence for all $g \in F$ (i.e., $g = 0$ on $[0, \frac{1}{2}]$), we have:

     $$\langle f, g\rangle = \int_0^1 f(t)g(t)\, dt = \int_0^{\frac{1}{2}} f(t)g(t)\, dt + \int_{\frac{1}{2}}^1 f(t)g(t)\, dt = 0$$

     thus $\{f \in E : f(t) = 0, \forall t \in [\frac{1}{2}, 1]\} \subset F^{\perp}$.

     Conversely, let $f \in F^{\perp}$; i.e., for all $g \in F$, we have $\langle f, g\rangle = 0$, hence: $\int_0^1 f(t)g(t)\, dt = 0$.

     In particular, taking $g(t) = \begin{cases} 0, & \text{if } t \in [0, \frac{1}{2}] \\ t - \frac{1}{2}, & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$

     clearly $g \in F$ so we can plug into the identity:

     $\int_0^1 f(t)g(t)\, dt = 0$, we get:

     $$\int_{\frac{1}{2}}^1 f(t)(t - \frac{1}{2})\, dt = 0$$

     which implies $f(t) = 0$, $\forall t \in [\frac{1}{2}, 1]$ (since f is continuous and positive.)

     Conclusion: $F^{\perp} = \{f \in E : f(t) = 0, \forall t \in [\frac{1}{2}, 1]\}$.

   - Yes; $F^{\perp\perp} = F$ because F is a closed subspace.

4. Let $h \equiv 1 \in E$, suppose $1 \in F + F^{\perp}$, then there exist $f \in F$ and $g \in F^{\perp}$ such that $1 = f + g$. We note that $f(\frac{1}{2}) = g(\frac{1}{2}) = 0 \neq 1$, which is a contradiction, so $E \neq F \oplus F^{\perp}$.

   We deduce that the assumption "F closed" is not sufficient for $E = F \oplus F^{\perp}$ when E is not complete. The correct assumption is that F must be complete.