Chapter 5

Correction of Exercises

5.1 Solutions to the Exercises of Chapter 1

Solution to Exercise 1.4

1. $\|\cdot\|_1$ is a norm, because:

   (a) Let $f \in E = C^1([0,1],\mathbb{R})$. If $\|f\|_1 = 0$, then:

$$\begin{aligned}
\|f\|_1 = 0 \quad &\Rightarrow \quad \sup_{x\in[0,1]} (|f(x)| + |f'(x)|) = 0 \\
&\Rightarrow \quad |f(x)| + |f'(x)| = 0 \\
&\Rightarrow \quad \begin{cases} |f(x)| = 0, & \forall x \in [0,1] \\ |f'(x)| = 0 & \forall x \in [0,1] \end{cases} \\
&\Rightarrow \quad |f(x)| = 0, \quad \forall x \in [0,1]
\end{aligned}$$

   Therefore, $f = 0$.

   (b) Let $f \in E$ and $\lambda \in \mathbb{R}$:

$$\begin{aligned}
\|\lambda f\|_1 &= \sup_{x\in[0,1]} (|\lambda f(x)| + |\lambda f'(x)|) \\
&= \sup_{x\in[0,1]} |\lambda(f(x)| + |f'(x)|) \\
&= |\lambda| \sup_{x\in[0,1]} |(f(x)| + |f'(x)|) = |\lambda| \|f\|_1.
\end{aligned}$$

   (c) Let $f, g \in E$; we have: $\|f+g\|_1 = \sup\limits_{x\in[0,1]} (|f(x)+g(x)| + |f'(x)+g'(x)|)$. For every $x \in [0,1]$, we have:

$$\begin{aligned}
|f(x)+g(x)| + |f'(x)+g'(x)| &\leq |f(x)| + |g(x)| + |f'(x)| + |g'(x)| \\
&\leq (|f(x)| + |f'(x)|) + (|g(x)| + |g'(x)|) \\
&\leq \sup_{x\in[0,1]} (|f(x)| + |f'(x)|) + \sup_{x\in[0,1]} (|g(x)| + |g'(x)|)
\end{aligned}$$

   and therefore:

$$\sup_{x\in[0,1]} |f(x)+g(x)| + |f'(x)+g'(x)| \leq \sup_{x\in[0,1]} (|f(x)| + |f'(x)|) + \sup_{x\in[0,1]} (|g(x)| + |g'(x)|),$$

   which shows that $\|f+g\|_1 \leq \|f\|_1 + \|g\|_1$. Consequently, $\|\cdot\|_1$ defines a norm on E.

2. On one hand, for all $x \in [0,1]$, we have:

$$|f(x)| + |f'(x)| \quad \leq \quad \sup_{x\in[0,1]} |f(x)| + \sup_{x\in[0,1]} |f'(x)|,$$

so

$$\sup_{x\in[0,1]} (|f(x)| + |f'(x)|) \leq \sup_{x\in[0,1]} |f(x)| + \sup_{x\in[0,1]} |f'(x)|,$$

hence: $\|f\|_1 \leq \|f\|_2$.

On the other hand, for all $x \in [0,1]$, we have:

$$\begin{aligned} |f(x)| &= |f(x)| + |f'(x)| - |f'(x)| \\ &\leq \sup_{x\in[0,1]} (|f(x)| + |f'(x)|) - |f'(x)|, \end{aligned}$$

thus

$$\sup_{x\in[0,1]} |f(x)| \leq \sup_{x\in[0,1]} (|f(x)| + |f'(x)|) - |f'(x)|$$

which implies:

$$|f'(x)| \leq \sup_{x\in[0,1]} (|f(x)| + |f'(x)|) - \sup_{x\in[0,1]} |f(x)|, \forall x \in [0,1],$$

therefore we get:

$$\sup_{x\in[0,1]} |f'(x)| \leq \sup_{x\in[0,1]} (|f(x)| + |f'(x)|) - \sup_{x\in[0,1]} |f(x)|,$$

hence:

$$\sup_{x\in[0,1]} |f(x)| + \sup_{x\in[0,1]} |f'(x)| \leq \sup_{x\in[0,1]} (|f(x)| + |f'(x)|), \forall x \in [0,1].$$

We deduce that $\|f\|_2 \leq \|f\|_1$. The conclusion is that the norms are equivalent ($\|f\|_2 \leq \|f\|_1 \leq \|f\|_2$) and furthermore $\|f\|_1 = \|f\|_2$.

Correction of Exercise 1.5:

1. $N(\cdot)$ is a norm; indeed:

   (a) Let $f \in E = C([0,1],\mathbb{R})$; if $N(f) = \int_0^1 x|f(x)|\ dx = 0$, since $x \mapsto x|f(x)|$ is a continuous function, then $x|f(x)| = 0$ for all $x \in [0,1]$, which implies $|f(x)| = 0$ for all $x \in]0,1]$, and by the continuity of f we have $f = 0$ on $[0,1]$.

   (b) Let $\lambda \in \mathbb{R}$ and $f \in E$, we have:

$$\begin{aligned} N(\lambda f) &= \int_0^1 x|\lambda f(x)|\ dx \\ &= |\lambda| \int_0^1 x|f(x)|\ dx = |\lambda| N(f). \end{aligned}$$

   (c) Let $f, g \in E$, we have:

$$\begin{aligned} N(f+g) &= \int_0^1 x|f(x)+g(x)|\ dx \\ &\leq \int_0^1 x(|f(x)| + |g(x)|)\ dx \\ &= \int_0^1 x|f(x)|\ dx + \int_0^1 x|g(x)|\ dx = N(f) + N(g). \end{aligned}$$

   We conclude that $N(\cdot)$ defines a norm on E.

2. We have:

$$\begin{aligned}
\|f_n\|_\infty &= \sup_{x\in[0,1]} |f_n(x)| \\
&= \max\left\{\sup_{x\in[0,\frac{1}{n}]} |f_n(x)|, \sup_{x\in[\frac{1}{n},1]} |f_n(x)| = 0\right\} \\
&= \max\left\{\sup_{x\in[0,\frac{1}{n}]} 1 - nx, 0\right\} = 1.
\end{aligned}$$

and

$$\begin{aligned}
N(f_n) &= \int_0^1 x|f_n(x)|\, dx \\
&= \int_0^{\frac{1}{n}} x|1-nx|\, dx + \int_{\frac{1}{n}}^1 x|0|\, dx \\
&= \int_0^{\frac{1}{n}} x - nx^2\, dx = \frac{1}{6n^2}.
\end{aligned}$$

Therefore, we have $\frac{\|f_n\|_\infty}{N(f_n)} = 6n^2 \to +\infty$ as $n \to +\infty$. We deduce that the norms $\|\cdot\|_\infty$ and $N(\cdot)$ are not equivalent.

Correction of Exercise 1.6

We consider the sequence of functions $f_n(t) = \sqrt{2n+1}\, t^n$, $t \in [0,1]$; we have:

$$\begin{aligned}
\|f_n\|_1 &= \int_0^1 \sqrt{2n+1}\, t^n\, dt = \frac{\sqrt{2n+1}}{n+1}, \\
\|f_n\|_2 &= \int_0^1 (2n+1)\, t^{2n}\, dt = 1, \\
\|f_n\|_\infty &= \sup_{t\in[0,1]} |\sqrt{2n+1}\, t^n| = \sqrt{2n+1}.
\end{aligned}$$

Hence:

$$\begin{aligned}
\lim_{n\to+\infty} \frac{\|f_n\|_1}{\|f_n\|_2} &= \lim_{n\to+\infty} \frac{\sqrt{2n+1}}{n+1} = +\infty, \\
\lim_{n\to+\infty} \frac{\|f_n\|_1}{\|f_n\|_\infty} &= \lim_{n\to+\infty} \frac{1}{n+1} = 0, \\
\lim_{n\to+\infty} \frac{\|f_n\|_2}{\|f_n\|_\infty} &= \lim_{n\to+\infty} \frac{1}{\sqrt{2n+1}} = 0,
\end{aligned}$$

Therefore, the three norms are not pairwise equivalent.

Correction of Exercise 1.7

1. (a) Clearly, φ is differentiable on $]0,+\infty[$ for every fixed $s \in]0,+\infty[$; moreover:

$$\varphi'(t) = t^{q-1} - s, \quad \forall t > 0,$$

We have

$$\varphi'(t) \geq 0 \Leftrightarrow t \geq s^{\frac{1}{q-1}},$$

Thus, φ is decreasing on $]0, s^{\frac{1}{q-1}}[$ and increasing on $]s^{\frac{1}{q-1}}, +\infty[$.

(b) From question 1.(a), we deduce that:

$$\varphi(t) \geq \varphi(s^{\frac{1}{q-1}}), \quad \forall t > 0,$$

but

$$\varphi(s^{\frac{1}{q-1}}) = \frac{s^p}{p} + \frac{s^{\frac{q}{q-1}}}{q} - s^{\frac{q}{q-1}},$$

since $\frac{1}{p} + \frac{1}{q} = 1$, we can write $p = \frac{q}{q-1}$, and therefore:

$$\varphi(s^{\frac{1}{q-1}}) = \frac{s^p}{p} + \frac{s^p}{q} - s^p = s^p \left(\frac{1}{p} + \frac{1}{q} - 1\right) = 0,$$

which shows that $\varphi(t) \geq 0$ for all $t > 0$, and thus for all $s, t \in]0, +\infty[$ we have:

$$st \leq \frac{s^p}{p} + \frac{t^q}{q}.$$

2. By taking

$$s = \frac{|x_k|}{\left(\sum_{k=1}^N |x_k|^p\right)^{1/p}}, \quad t = \frac{|y_k|}{\left(\sum_{k=1}^N |y_k|^q\right)^{1/q}},$$

for all $k = 1, 2, ..., N$, and for simplicity denote:

$$a = \left(\sum_{k=1}^N |x_k|^p\right)^{1/p}, \quad b = \left(\sum_{k=1}^N |y_k|^q\right)^{1/q}.$$

Then, we obtain:

$$\frac{|x_k y_k|}{ab} \leq \frac{|x_k|^p}{pa^p} + \frac{|y_k|^q}{qb^q}, \quad \forall k = 1, 2, ..., N.$$

Hence:

$$\sum_{k=1}^N \frac{|x_k y_k|}{ab} \leq \sum_{k=1}^N \frac{|x_k|^p}{pa^p} + \sum_{k=1}^N \frac{|y_k|^q}{qb^q} = \frac{a^p}{pa^p} + \frac{b^q}{qb^q} = 1,$$

thus:

$$\sum_{k=1}^N |x_k y_k| \leq ab = \left(\sum_{k=1}^N |x_k|^p\right)^{1/p} \left(\sum_{k=1}^N |y_k|^q\right)^{1/q},$$

and don't forget that:

$$\left|\sum_{k=1}^N x_k y_k\right| \leq \sum_{k=1}^N |x_k y_k|.$$

So we obtain:

$$\left|\sum_{k=1}^N x_k y_k\right| \leq \left(\sum_{k=1}^N |x_k|^p\right)^{1/p} \left(\sum_{k=1}^N |y_k|^q\right)^{1/q}.$$

3. (a) Write $|x_k + y_k|^p = |x_k + y_k| \cdot |x_k + y_k|^{p-1}$, and using the triangle inequality:

$$\begin{aligned}
\sum_{k=1}^N |x_k + y_k|^p &= \sum_{k=1}^N |x_k + y_k| \cdot |x_k + y_k|^{p-1} \\
&\leq \sum_{k=1}^N (|x_k| + |y_k|) \cdot |x_k + y_k|^{p-1} \\
&= \sum_{k=1}^N |x_k| \cdot |x_k + y_k|^{p-1} + \sum_{k=1}^N |y_k| \cdot |x_k + y_k|^{p-1}.
\end{aligned}$$

(b) By Hölder's inequality:

$$\begin{aligned}
\sum_{k=1}^{N} |x_k| \cdot |x_k + y_k|^{p-1} &= \left| \sum_{k=1}^{N} |x_k| \cdot |x_k + y_k|^{p-1} \right| \\
&\leq \left(\sum_{k=1}^{N} |x_k|^p \right)^{1/p} \left(\sum_{k=1}^{N} |x_k + y_k|^{q(p-1)} \right)^{1/q} \\
&= \left(\sum_{k=1}^{N} |x_k|^p \right)^{1/p} \left(\sum_{k=1}^{N} |x_k + y_k|^p \right)^{1/q} \qquad (5.1)
\end{aligned}$$

and similarly:

$$\sum_{k=1}^{N} |y_k| \cdot |x_k + y_k|^{p-1} \leq \left(\sum_{k=1}^{N} |y_k|^p \right)^{1/p} \left(\sum_{k=1}^{N} |x_k + y_k|^p \right)^{1/q}. \qquad (5.2)$$

Using 3.(a), (5.1) and (5.2), we obtain:

$$\begin{aligned}
\sum_{k=1}^{N} |x_k + y_k|^p &\leq \sum_{k=1}^{N} |x_k| \cdot |x_k + y_k|^{p-1} + \sum_{k=1}^{N} |y_k| \cdot |x_k + y_k|^{p-1} \\
&\leq \left[\left(\sum_{k=1}^{N} |x_k|^p \right)^{1/p} + \left(\sum_{k=1}^{N} |y_k|^p \right)^{1/p} \right] \left(\sum_{k=1}^{N} |x_k + y_k|^p \right)^{1/q}
\end{aligned}$$

hence:

$$\left(\sum_{k=1}^{N} |x_k + y_k|^p \right)^{1-\frac{1}{q}} \leq \left(\sum_{k=1}^{N} |x_k|^p \right)^{1/p} + \left(\sum_{k=1}^{N} |y_k|^p \right)^{1/p},$$

and consequently:

$$\left(\sum_{k=1}^{N} |x_k + y_k|^p \right)^{1/p} \leq \left(\sum_{k=1}^{N} |x_k|^p \right)^{1/p} + \left(\sum_{k=1}^{N} |y_k|^p \right)^{1/p}.$$

4. $\| \cdot \|$ is a norm; indeed, the first two conditions are obvious. The third condition follows directly from Minkowski's inequality.

5. It suffices to take the limit.

Solution to Exercise ??

1. Obvious.

2. Equivalence of the norms:

   - On one hand, we observe that:

$$|f(0)| \leq \sup_{x \in [0,1]} |f(x)|,$$

thus,

$$\|f'\|_\infty + |f(0)| \leq \|f'\|_\infty + \sup_{x \in [0,1]} |f(x)|,$$

i.e.,

$$\|f\|_1 \leq \|f\|.$$

- On the other hand, knowing that:

$$f(x) = f(0) + \int_0^x f'(t)\, dt,$$

then, for all $x \in [0,1]$, we have:

$$\begin{aligned} |f(x)| &\leq |f(0)| + \left| \int_0^x f'(t)\, dt \right| \\ &\leq |f(0)| + \int_0^x |f'(t)|\, dt \\ &\leq |f(0)| + \int_0^1 |f'(t)|\, dt \\ &\leq |f(0)| + \sup_{t\in[0,1]} |f'(t)| \int_0^1 dt \\ &= |f(0)| + \|f'\|_\infty, \end{aligned}$$

hence $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)| \leq |f(0)| + \|f'\|_\infty$ and therefore, we have:

$$\begin{aligned} \|f\| = \|f\|_\infty + \|f'\|_\infty &\leq |f(0)| + 2\|f'\|_\infty \\ &\leq 2(|f(0)| + \|f'\|_\infty) = 2\|f\|_1. \end{aligned}$$

We have shown that

$$\frac{1}{2}\|f\| \leq \|f\|_1 \leq \|f\|.$$

3. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(E, \|\cdot\|_1)$; that is,

(a)

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall p, q \in \mathbb{N} : p > q \geq n_0 \Rightarrow \|f_p - f_q\|_1 \leq \varepsilon;$$

so we have:

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall p, q \in \mathbb{N} : p > q \geq n_0 \Rightarrow \|f'_p - f'_q\|_\infty + |f_p(0) - f_q(0)| \leq \varepsilon;$$

hence:

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall p, q \in \mathbb{N} : p > q \geq n_0 \Rightarrow \begin{cases} \|f'_p - f'_q\|_\infty \leq \varepsilon & (*) \\ \wedge & \\ |f_p(0) - f_q(0)| \leq \varepsilon, & (**) \end{cases}$$

(b) From (*) and since the sequence $(f'_n) \subset F$, it follows that it is Cauchy in $(F, \|\cdot\|_\infty)$, which is complete, hence (f'_n) converges in F to a limit denoted g, i.e.,

$$\lim_{n\to+\infty} \|f'_n - g\|_\infty = 0.$$

From (**) and since the sequence $(f_n(0)) \subset \mathbb{R}$, it follows that it is Cauchy in $(\mathbb{R}, |\cdot|)$, which is complete, so $(f_n(0))$ converges in $\mathbb{R}$ to a limit denoted l, i.e.,

$$\lim_{n\to+\infty} |f_n(0) - l| = 0.$$

(c) First, note that $f \in E$ and $f' = g$ and $f(0) = l$, moreover:

$$\begin{aligned} \|f_n - f\|_1 &= \|f_n' - f'\|_\infty + |f_n(0) - f(0)| \\ &= \|f_n' - g\|_\infty + |f_n(0) - l| \\ &\to 0 \quad n \to +\infty \end{aligned}$$

therefore f_n converges to f in $(E, \|\cdot\|_1)$.

4. The space $(E, \|\cdot\|_1)$ is Banach, and since $\|\cdot\|_1$ and $\|\cdot\|$ are equivalent norms, then $(E, \|\cdot\|)$ is also Banach.

Solution to Exercise 1.9

1. First, note that the functions u_n are continuous for all $n \in \mathbb{N}$, so $(u_n) \subset E$. It is a Cauchy sequence. Indeed, let $\varepsilon > 0$ and let $p, q \in \mathbb{N}$ with $p > q$, we estimate $\|u_p - u_q\|$. We rely on the following table:

x	-1	$\frac{-1}{q}$	$\frac{-1}{p}$	0	1
$u_p(x)$	0	0	$px+1$	1	
$u_q(x)$	0	$qx+1$	$qx+1$	1	
$u_p(x) - u_q(x)$	0	$-(qx+1)$	$(p-q)x$	0	

Table 5.1: Table to test captions and labels

thus:

$$\begin{aligned} \|u_p - u_q\|^2 &= \int_{-1}^{1} |u_p - u_q|^2 \, dx \\ &= \int_{-\frac{1}{q}}^{-\frac{1}{p}} |qx+1|^2 \, dx + (p-q)^2 \int_{-\frac{1}{p}}^{0} x^2 \, dx \\ &= \frac{(p-q)^3}{3qp^3} + \frac{(p-q)^2}{3p^3} \\ &\leq \frac{2}{q} \to 0 \quad \text{as } q \to +\infty. \end{aligned}$$

2. Moreover, it is clear that the sequence (u_n) converges pointwise to the function u with:

$$u(x) = \begin{cases} 0, & \text{if } -1 \leq x < 0; \\ 1, & \text{if } 0 \leq x \leq 1. \end{cases}$$

We have:

$$\begin{aligned} \|u_n - u\|^2 &= \int_{-1}^{-\frac{1}{n}} (nx+1)^2 \, dx \\ &= \frac{1}{3n} \to 0 \quad \text{as } n \to +\infty. \end{aligned}$$

Finally, we observe that u is not continuous at 0, so it does not belong to E and hence (u_n) does not converge in E.

3. $(E, \|\cdot\|)$ is not complete and thus not a Banach space.

Correction of Exercise 1.11

We observe that E is finite-dimensional, hence all norms are equivalent, and both $\int_0^1 |P(x)|\, dx$ and $\sup_{x\in[0,1]} |P(x)|$ define norms on E. Therefore, they are equivalent, which gives the result.

Correction of Exercise 1.12

Since E is finite-dimensional, it suffices to show that M is closed and bounded. First, we show that M is bounded. Indeed, since K is compact, it is bounded, thus:

$$\exists c > 0, \forall x \in K : \|x\| \leq c,$$

Now take $y \in M$, then there exists $x \in K$ such that $y \in \overline{B}(x, r)$, i.e.,

$$\|y - x\| \leq r,$$

which gives:

$$\|y\| = \|y - x + x\| \leq \|y - x\| + \|x\| \leq r + c,$$

and therefore M is bounded.

Now we show that M is closed. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in M that converges to $y \in E$. We want to prove that $y \in M$. For every $n \in \mathbb{N}$, we have $y_n \in M$, i.e.,

$$\forall n \in \mathbb{N}, \exists x_n \in K : \|y_n - x_n\| \leq r,$$

Since K is compact, there exists $x \in K$ and a subsequence (x_{n_k}) converging to x. Also, the subsequence (y_{n_k}) converges to y. We have

$$\|y_{n_k} - x_{n_k}\| \leq r,$$

Taking the limit yields:

$$\|y - x\| \leq r,$$

which shows that $y \in M$, hence M is compact.

Correction of Exercise 1.13

Let $n, q \in \mathbb{N}$ with $n \neq q$:

1.

$$\begin{aligned}
\|f_n - f_q\|_2^2 &= \int_0^{2\pi} (\cos px)^2 + (\cos qx)^2 - 2\cos px \cos qx \; dx \\
&= \frac{1}{2}\int_0^{2\pi} (\cos 2px + \cos 2qx + 2)\; dx - \int_0^{2\pi} (\cos(p+q)x + \cos(p-q)x)\; dx \\
&= 2\pi.
\end{aligned}$$

Hence $\|f_n - f_q\|_2 = \sqrt{2\pi}$.

$$\|f_n\|_2^2 = \int_0^{2\pi} (\cos nx)^2\; dx = \frac{1}{2}\int_0^{2\pi} (\cos 2nx + 1)\; dx = \pi \Rightarrow \|f_n\|_2 = \sqrt{\pi}.$$

2. It is evident that $(g_n) \subset E$, and moreover, for all $n, q \in \mathbb{N}$ with $n \neq q$, we have:
$$\|g_n - g_q\|_2 = \frac{1}{\|f_n\|_2}\|f_n - f_q\|_2 = \frac{1}{\sqrt{\pi}}\sqrt{2\pi} = \sqrt{2}.$$
Suppose that (g_n) admits a convergent subsequence (g_{n_k}), then:
$$\|g_{n_{k+1}} - g_{n_k}\|_2$$
should tend to 0, which is not the case since $\|g_{n_{k+1}} - g_{n_k}\|_2 = \sqrt{2}$. It follows that the sequence (g_n) cannot have any convergent subsequence.

3. We have:
$$\|g_n\|_2 = \left\|\frac{f_n}{\|f_n\|_2}\right\|_2 = \frac{1}{\|f_n\|_2}\|f_n\|_2 = 1,$$
Hence, $(g_n)_n \subset \overline{B}(0,1)$. From question 3, we conclude that there exists in $\overline{B}(0,1)$ a sequence that does not admit any convergent subsequence. Therefore, the closed unit ball is not compact.

4. Since the closed unit ball is not compact, then by F. Riesz's theorem, we deduce that E is infinite-dimensional.

5.2 Solutions to the Exercises of Chapter 2

Solution to Exercise 2.3:

1. (a) $\|f_n\|_1 = \sqrt{n}\int_0^1 x^n\,dx = \frac{\sqrt{n}}{n+1}$.

   (b) T is not continuous. Indeed, we have:
$$\frac{|T(f_n)|}{\|f_n\|_1} = \frac{f_n(1)}{\|f_n\|_1} = \frac{\sqrt{n}}{\frac{\sqrt{n}}{n+1}} = n+1,$$
   thus $\lim_{n\to+\infty} \frac{|T(f_n)|}{\|f_n\|_1} = \lim_{n\to+\infty} n+1 = +\infty$, which implies that T is not continuous.

2. Let $f \in E$, we have: $|T(f)| = |f(1)| \leq \sup_{x\in[a,b]} |f(x)| = \|f\|_\infty$. Hence, T is continuous.

Solution to Exercise 2.5:

- Norm of the identity map from $(C^1([0,1],\mathbb{R}), \|\cdot\|)$ into $(C^1([0,1],\mathbb{R}), \|\cdot\|_1)$. In Exercise 1.8, we proved that:
$$\frac{1}{2}\|f\| \leq^1 \|f\|_1 \leq^2 \|f\|.$$
From inequality $\leq^2$, we obtain $\|id\|_{\mathcal{L}(E)} \leq 1$. On the other hand, take $f_0 = 1 \in E$; it satisfies $\|f_0\| = 1$ and $\|id(f_0)\|_1 = \|f_0\|_1 = \|f_0'\|_\infty + \|f_0\|_\infty = 1$. Thus $\|id\|_{\mathcal{L}(E)} \geq 1$ and consequently $\|id\|_{\mathcal{L}(E)} = 1$.

- Norm of the identity map from $(C^1([0,1],\mathbb{R}), \|\cdot\|_1)$ into $(C^1([0,1],\mathbb{R}), \|\cdot\|)$. From inequality $\leq^1$, we have $\|id\|_{\mathcal{L}(E)} \leq 2$. On the other hand, take $f_1(x) = x$; it satisfies $f_1 \in E$, $\|f_1\|_1 = 1$, and $\|id(f_1)\| = \|f_1\| = \|f_1'\|_\infty + \|f_1\|_\infty = 2$. Thus $\|id\|_{\mathcal{L}(E)} \geq 2$ and hence $\|id\|_{\mathcal{L}(E)} = 2$.

Solution to Exercise 2.9:

1. Clearly, ϕ is linear. Let's show that ϕ is continuous. Let $f \in E$ and $x \in [a,b]$, we have:

$$\begin{aligned}|\phi(f)(x)| &= \left|\int_a^x f(t)p(t)\,dt\right| \\ &\leq \int_a^x |f(t)p(t)|\,dt \\ &\leq \int_a^b |f(t)p(t)|\,dt \\ &\leq \sup_{t\in[a,b]} |f(t)| \int_a^b |p(t)|\,dt = c\|f\|_\infty,\end{aligned}$$

where $c = \int_a^b |p(t)|\,dt$. Therefore,

$$\|\phi(f)\|_\infty = \sup_{x\in[a,b]} |\phi(f)(x)| \leq c\|f\|_\infty, \tag{5.3}$$

which shows that ϕ is continuous.

2. By definition,

$$\|\phi\|_{\mathcal{L}(E)} = \sup_{\|f\|_\infty \leq 1} \|\phi(f)\|_\infty,$$

so from equation (5.3), we get:

$$\|\phi\|_{\mathcal{L}(E)} \leq \int_a^b |p(t)|\,dt. \tag{5.4}$$

3. First, we observe that:

$$\begin{aligned}\|\phi\|_{\mathcal{L}(E)} \geq \int_a^b p(t)\,dt \quad &\Leftrightarrow \quad \sup_{\|f\|_\infty \leq 1} \|\phi(f)\|_\infty \geq \int_a^b p(t)\,dt \\ &\Leftarrow \quad \exists f_0 \in E : \|f_0\|_\infty \leq 1 \wedge \|\phi(f_0)\|_\infty \geq \int_a^b p(t)\,dt \\ &\Leftrightarrow \quad \exists f_0 \in E : \|f_0\|_\infty \leq 1 \wedge \sup_{x\in[a,b]} |\phi(f_0)(x)| \geq \int_a^b p(t)\,dt \\ &\Leftarrow \quad \exists f_0 \in E : \|f_0\|_\infty \leq 1, \quad \exists x_0 \in [a,b] : |\phi(f_0)(x_0)| \geq \int_a^b p(t)\,dt.\end{aligned}$$

So if we take $f_0 = 1 \in E$ and $x_0 = b \in [a,b]$, we obtain:

$$|\phi(f_0)(x_0)| = \int_a^b p(t)\,dt \geq \int_a^b p(t)\,dt. \tag{5.5}$$

Combining (5.4) and (5.5), we get:

$$\|\phi\|_{\mathcal{L}(E)} = \int_a^b p(t)\,dt.$$

4. (a) Clearly, $(f_n)_n \subset E$. Let us compute $\|\phi(f_n)\|_\infty$:

$$\begin{aligned}\|\phi(f_n)\|_\infty &= \sup_{x\in[a,b]} |\phi(f_n)(x)| \\ &= \sup_{x\in[a,b]} \left|\int_a^x \frac{n(p(t))^2}{\sqrt{1+n^2(p(t))^2}}\,dt\right| \\ &= \int_a^b \frac{n(p(t))^2}{\sqrt{1+n^2(p(t))^2}}\,dt.\end{aligned}$$

(b) For all $t \in [a,b]$, we have:

$$\lim_{n\to+\infty} \frac{n(p(t))^2}{\sqrt{1+n^2(p(t))^2}} = |p(t)|,$$

and

$$\frac{n(p(t))^2}{\sqrt{1+n^2(p(t))^2}} \leq \frac{n(p(t))^2}{\sqrt{n^2(p(t))^2}} = |p(t)|,$$

hence by the Dominated Convergence Theorem:

$$\lim_{n\to+\infty} \|\phi(f_n)\|_\infty = \lim_{n\to+\infty} \int_a^b \frac{n(p(t))^2}{\sqrt{1+n^2(p(t))^2}}\, dt = \int_a^b |p(t)|\, dt.$$

which means that

$$\|\phi\|_{\mathcal{L}(E)} \geq \int_a^b |p(t)|\, dt. \tag{5.6}$$

From (5.4) and (5.6), we conclude that:

$$\|\phi\|_{\mathcal{L}(E)} = \int_a^b |p(t)|\, dt.$$

Solution to Exercise 2.10:
We write:

$$\begin{aligned} F &= \{f \in E : f(0) = 0 \text{ and } \int_0^1 f(t)\, dt \geq 1\} \\ &= \{f \in E : f(0) = 0\} \cap \{f \in E : \int_0^1 f(t)\, dt \geq 1\}. \end{aligned}$$

We consider the following applications:

$$\begin{aligned} T_1 : (E, \|\cdot\|_\infty) &\to \mathbb{R} \\ f &\mapsto T_1(f) = f(0), \end{aligned}$$

and

$$\begin{aligned} T_2 : (E, \|\cdot\|_\infty) &\to \mathbb{R} \\ f &\mapsto T_2(f) = \int_0^1 f(t)\, dt. \end{aligned}$$

It is easy to see that T_1 and T_2 are linear and continuous mappings. Moreover:

$$F = T_1^{-1}\Big(\{0\}\Big) \cap T_2^{-1}\Big([1,+\infty[\Big).$$

Since the sets $\{0\}$ and $[1,+\infty[$ are closed in $\mathbb{R}$, it follows that $T_1^{-1}\Big(\{0\}\Big)$ and $T_2^{-1}\Big([1,+\infty[\Big)$ are closed in E as preimages of closed sets under continuous mappings. Their intersection is also closed, and thus F is a closed set. **Correction of Exercise** 2.14:

1. Let $h = (h_n) \in H$, we have:

$$|\varphi(x)| = |\varphi(x) - \varphi(h)| = |\varphi(x-h)| \leq \|\varphi\|_{E'}\|x-h\|,$$

hence

$$\frac{|\varphi(x)|}{\|\varphi\|_{E'}} \leq \inf_{h\in H} \|x-h\| = d(x,H). \tag{5.7}$$

2. Let $b \in E \setminus H$ and let $x \in E$. Writing $x = \frac{\varphi(x)}{\varphi(b)}b + x - \frac{\varphi(x)}{\varphi(b)}b$ and noting that

$$\varphi\Big(x - \frac{\varphi(x)}{\varphi(b)}b\Big) = 0,$$

therefore $x - \frac{\varphi(x)}{\varphi(b)}b \in H$. On the other hand, by definition we have:

$$\|\varphi\|_{E'} = \sup_{\|x\|=1} |\varphi(x)|,$$

thus, there exists $n_0 \in \mathbb{N}^*$ such that for all $n \geq n_0$, there exists $x_n \in E$ such that $\|x_n\| = 1$ and:

$$0 < \|\varphi\|_{E'} - \frac{1}{n} < |\varphi(x_n)| \leq \|\varphi\|_{E'}$$

so $x_n \notin H$. We take $b = x_n$, and $t_n = \frac{\varphi(x)}{\varphi(x_n)} \in \mathbb{R}$, and $h_n = x - \frac{\varphi(x)}{\varphi(x_n)}x_n \in H$. Therefore, we can write:

$$x = h_n + t_n x_n.$$

3. We know that $t_n = \frac{\varphi(x)}{\varphi(x_n)}$, so:

$$|t_n| = \frac{|\varphi(x)|}{|\varphi(x_n)|} \leq \frac{|\varphi(x)|}{\|\varphi\|_{E'} - \frac{1}{n}}.$$

Knowing that $x - h_n = t_n x_n$, we have:

$$\|x - h_n\| = \|t_n x_n\| = |t_n| \leq \frac{|\varphi(x)|}{\|\varphi\|_{E'} - \frac{1}{n}},$$

hence we deduce that

$$d(x, H) \leq \frac{|\varphi(x)|}{\|\varphi\|_{E'}}. \tag{5.8}$$

From (5.7) and (5.8), we conclude:

$$d(x, H) = \frac{|\varphi(x)|}{\|\varphi\|_{E'}}.$$

Correction of Exercise 2.15:

1. It is clear that ϕ is linear. Let $x = (x_n)_{n\in\mathbb{N}} \in \ell_0$, we have:

$$|\phi(x)| \leq \sum_{n=0}^{\infty} \frac{|x_n|}{2^n} \leq \sup_{n\in\mathbb{N}} |x_n| \sum_{n=0}^{\infty} \frac{1}{2^n} = 2\|x\|,$$

hence ϕ is continuous. We also deduce that $\|\phi\|_{\ell_0'} \leq 2$. On the other hand, consider the following sequence:
$X_N = (x_n)$ with $x_n = 1$ if $0 \leq n \leq N$, and $x_n = 0$ otherwise. Then we have $\|X_N\| = 1$, and also:

$$\phi(X_N) = \sum_{n=0}^{N} \frac{1}{2^n} = 2\left(1 - \frac{1}{2^{N+1}}\right).$$

So we get $\|\phi\|_{\ell_0'} \geq 2\left(1 - \frac{1}{2^{N+1}}\right)$, and by letting $N \to +\infty$, we obtain $\|\phi\|_{\ell_0'} \geq 2$. Consequently, $\|\phi\|_{\ell_0'} = 2$.

2. Let $h = (h_n)_{n\in\mathbb{N}} \in H$, then we have $\varphi(h) := \sum_{n=0}^{\infty} \frac{h_n}{2^n} = 0$, hence $\frac{|\varphi(x)|}{\|\varphi\|_{\ell_0'}} := \sum_{n=0}^{\infty} \frac{h_n}{2^n+1} = 0$. Let $x = (x_n)_{n\in\mathbb{N}} \in E \setminus H$, then from exercise 2.15 we have $d(x, H) = \frac{|\varphi(x)|}{\|\varphi\|_{\ell_0'}}$. Moreover:

$$\begin{aligned}
d(x,H) = \frac{|\varphi(x)|}{\|\varphi\|_{\ell_0'}} &= \Big| \sum_{n=0}^{\infty} \frac{x_n}{2^{n+1}} \Big| \\
&= \Big| \sum_{n=0}^{\infty} \frac{x_n}{2^{n+1}} - \sum_{n=0}^{\infty} \frac{h_n}{2^{n+1}} \Big| \\
&\leq \sum_{n=0}^{\infty} \frac{|x_n - h_n|}{2^{n+1}} \\
&\leq \sum_{n=0}^{\infty} \frac{\|x-h\|_\infty}{2^{n+1}} \\
&= \|x-h\|_\infty \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} = \|x-h\|_\infty.
\end{aligned}$$

If we suppose that $d(x, H) = d(x, h) = \|x - h\|_\infty$, then we have:

$$\sum_{n=0}^{\infty} \frac{|x_n - h_n|}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{\|x-h\|_\infty}{2^{n+1}}.$$

But for all $n \in \mathbb{N}$, we have:

$$|x_n - h_n| \leq \|x-h\|_\infty,$$

hence we deduce that for all $n \in \mathbb{N}$, $|x_n - h_n| = \|x - h\|_\infty$. Since:

$$\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} h_n = 0,$$

we deduce that $x = h$, which is a contradiction. Therefore, there is no $h \in H$ such that $d(x, H) = d(x, h) = \|x - h\|_\infty$.

5.3 Solutions to Chapter 3 Exercises

Correction of Exercise 3.9

1. This is obvious for the first part.

2. Let $A = (a_{ij})_{1\leq i,j\leq n}$ and $B = (b_{ij})_{1\leq i,j\leq n}$. We have:

$$\operatorname{tr}(A^T B) = \sum_{k=1}^{n}\sum_{i=1}^{n} a_{i,k} b_{i,k}. \tag{5.9}$$

- Let $A, A', B \in \mathcal{M}_n(\mathbb{R})$ and $\lambda \in \mathbb{R}$, we have:

$$\begin{aligned}
\langle A + \lambda A', B\rangle &= \operatorname{tr}((A+\lambda A')^T B) \\
&= \sum_{k=1}^{n}\sum_{i=1}^{n} (a_{i,k} + \lambda a'_{i,k}) b_{i,k} \\
&= \sum_{k=1}^{n}\sum_{i=1}^{n} a_{i,k} b_{i,k} + \lambda \sum_{k=1}^{n}\sum_{i=1}^{n} a'_{i,k} b_{i,k} \\
&= \langle A, B\rangle + \langle A', B\rangle.
\end{aligned}$$

- $\langle \cdot, \cdot \rangle$ is symmetric, since:
$$\langle A, B\rangle = \sum_{k=1}^{n}\sum_{i=1}^{n} a_{i,k}b_{i,k} = \sum_{k=1}^{n}\sum_{i=1}^{n} b_{i,k}a_{i,k} = \langle B, A\rangle.$$
- For all $A = (a_{ij}) \in \mathcal{M}_n(\mathbb{R})$, we have:
$$\langle A, A\rangle = \sum_{k=1}^{n}\sum_{i=1}^{n} a_{i,k}^2 \geq 0,$$
and:
$$\langle A, A\rangle = 0 \Leftrightarrow a_{i,k}^2 = 0, \quad \forall i, k \in \{1, ..., n\},$$
which means that A is the zero matrix. Therefore, $\langle \cdot, \cdot \rangle$ defines an inner product.

Correction of Exercise 3.11 Just apply Theorem 2.2 and use the Cauchy-Schwarz inequality.

Correction of Exercise 3.14

1. Suppose that $\|x + y\| = \|x\|^2 + \|y\|$, then $\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\|x\|\|y\|$. On the other hand:
$$\|x + y\|^2 = \langle x + y, x + y\rangle = \langle x, x\rangle + \langle y, y\rangle + 2\langle x, y\rangle.$$
By comparison, we get $\langle x, y\rangle = \|x\|\|y\|$.

2. (a) By direct calculation:
$$\begin{aligned}\left\|x - \frac{\langle x, y\rangle}{\|y\|^2}y\right\|^2 &= \langle x - \frac{\langle x, y\rangle}{\|y\|^2}y, x - \frac{\langle x, y\rangle}{\|y\|^2}y\rangle \\ &= \|x\|^2 - \frac{|\langle x, y\rangle|^2}{\|y\|^2}.\end{aligned}$$
(b) Suppose $\langle x, y\rangle = \|x\|\|y\|$, then:
$$\left\|x - \frac{\langle x, y\rangle}{\|y\|^2}y\right\|^2 = \|x\|^2 - \frac{\|x\|^2\|y\|^2}{\|y\|^2} = 0,$$
which means $x - \frac{\langle x,y\rangle}{\|y\|^2}y = 0$. If $x = 0$ or $y = 0$, just take $a = b = 0$. Otherwise, assume $x, y \neq 0$ and let:
$$\lambda = \frac{\langle x, y\rangle}{\|y\|^2} = \frac{\|x\|\|y\|}{\|y\|^2} = \frac{\|x\|}{\|y\|} \geq 0,$$
then $x = \lambda y$, and we can take $a = 1$ and $b = \lambda$.

3. Suppose there exist $a, b \geq 0$ with $(a, b) \neq (0, 0)$ such that $ax = by$. If $x = 0$ or $y = 0$, the implication is immediate. Otherwise, $x \neq 0$ and $y \neq 0$ implies $a, b > 0$. Then:
$$\|x + y\| = \|x + \frac{a}{b}x\| = \left(1 + \frac{a}{b}\right)\|x\| = \|x\| + \left\|\frac{a}{b}x\right\| = \|x\| + \|y\|,$$
which concludes the proof of the equivalences $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$.

Correction of Exercise 3.15 It suffices to check that the functions $\cos t$ and $\sin t$ do not satisfy the parallelogram identity.

Correction of Exercise 3.16 The norm associated with the inner product is $\|u\|_2 = \left(\int_{-1}^{1} |u(t)|^2\, dt\right)^{\frac{1}{2}}$. Exercise 1.9 shows that $(E, \|\cdot\|_2)$ is a normed space but not complete, which implies that $(E, \langle\cdot,\cdot\rangle)$ is a pre-Hilbert space but not a Hilbert space.

Correction of Exercise 3.19

1. Let $f, g \in H$. We have:
$$\frac{1}{\sqrt{1-x^2}} f(x)g(x) = \frac{1}{\sqrt{1-x}} \frac{1}{\sqrt{1+x}} f(x)g(x).$$
The function $x \mapsto \frac{1}{\sqrt{1-x^2}} f(x)g(x)$ is continuous on $]-1,1[$. Moreover, the function $x \mapsto \frac{1}{\sqrt{1+x}} f(x)g(x)$ is continuous at 1, so it is bounded in a neighborhood of 1. Fix $\varepsilon \in]-1,1[$, we have:
$$\begin{aligned} \left|\int_{\varepsilon}^{1} \frac{1}{\sqrt{1-x^2}} f(x)g(x)\, dx\right| &= \int_{\varepsilon}^{1} \frac{1}{\sqrt{1-x}} \left|\frac{1}{\sqrt{1+x}} f(x)g(x)\right| dx \\ &\le C \int_{\varepsilon}^{1} \frac{1}{\sqrt{1-x}}\, dx = 2C\sqrt{1-\varepsilon} < \infty. \end{aligned}$$
Similarly, the function $x \mapsto \frac{1}{\sqrt{1-x}} f(x)g(x)$ is continuous at -1, hence bounded in a neighborhood of -1. Then we have:
$$\begin{aligned} \left|\int_{-1}^{\varepsilon} \frac{1}{\sqrt{1-x^2}} f(x)g(x)\, dx\right| &= \int_{-1}^{\varepsilon} \frac{1}{\sqrt{1+x}} \left|\frac{1}{\sqrt{1-x}} f(x)g(x)\right| dx \\ &\le C \int_{-1}^{\varepsilon} \frac{1}{\sqrt{1+x}}\, dx = 2C\sqrt{1+\varepsilon} < \infty. \end{aligned}$$
Finally, the function $x \mapsto \frac{1}{\sqrt{1-x^2}} f(x)g(x)$ is integrable on $]-1,1[$.

   Now we verify that $\langle\cdot,\cdot\rangle$ defined in (3.23) is an inner product. Symmetry, bilinearity, and positivity are clear. It remains to show that $\langle\cdot,\cdot\rangle$ is definite. For all $f \in H$, we have:
$$\begin{aligned} \langle f, f\rangle = 0 \;&\Leftrightarrow\; \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f^2(x)\, dx = 0 \\ &\Leftrightarrow\; \frac{1}{\sqrt{1-x^2}} f^2(x) = 0, \forall x \in]-1,1[\\ &\;\text{(since the function is continuous, positive, and has zero integral)} \\ &\Leftrightarrow\; f(x) = 0, \forall x \in]-1,1[\Leftrightarrow f = 0. \end{aligned}$$
Hence, $\langle\cdot,\cdot\rangle$ is indeed an inner product.

2. Let $n, m \in \mathbb{N}$ with $n \neq m$. We have:
$$\langle T_n, T_m\rangle = \frac{2}{\pi} \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \cos(n \arccos x) \sin(n \arcsin x)\, dx,$$
Set $x = \cos t$, then $dx = -\sin t dt$, we get:
$$\begin{aligned} \langle T_n, T_m\rangle &= \frac{2}{\pi} \int_{\pi}^{0} \frac{1}{\sqrt{1-\cos^2}} \cos(nt) \sin(nt)(-\sin t)\, dt \\ &= \int_{0}^{\pi} \cos(nt) \sin(nt)\, dt, \end{aligned}$$

since $\cos(nt)\sin(nt) = \frac{1}{2}[\cos((n-m)t) + \cos((n+m)t)]$, we have:

$$\begin{aligned}
\langle T_n, T_m \rangle &= \frac{1}{\pi}\int_0^{\pi} \cos((n-m)t) + \cos((n+m)t)\, dt \\
&= \frac{1}{\pi}\Big[\frac{1}{n-m}\sin((n-m)t) + \frac{1}{n+m}\sin((n+m)t)\Big]_0^{\pi} \\
&= 0.
\end{aligned}$$

Thus, the family $(T_n)_{n\in\mathbb{N}}$ is orthogonal.

Furthermore, for $n = m \in \mathbb{N}^*$, we have:

$$\langle T_n, T_n \rangle = \frac{1}{\pi}\int_0^{\pi} 1 + \cos(2nt)\, dt = \frac{1}{\pi}[t + \frac{1}{2n}\sin 2nt]_0^{\pi} = 1.$$

If $n = m = 0$, then $\langle T_0, T_0 \rangle = 2$. We deduce that the family $(T_n)_{n\in\mathbb{N}^*}$ is orthonormal.

Correction of Exercise 3.21

1. (a) Linearity is clear. For all $x = (x_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$, by the Cauchy-Schwarz inequality we have:

$$\begin{aligned}
J(x) = \Big|\sum_{n=0}^{N} x_n\Big| &\leq \Big(\sum_{n=0}^{N} 1\Big)^{1/2}\Big(\sum_{n=0}^{N} x_n^2\Big)^{1/2} \\
&\leq \sqrt{N+1}\Big(\sum_{n=0}^{\infty} x_n^2\Big)^{1/2} = C\|x\|.
\end{aligned}$$

Hence, J is continuous.

(b) We observe that $F_N = J^{-1}(\{0\})$, so we deduce that F_N is closed. Moreover, it is easy to see that F_N is a subspace of $\ell^2(\mathbb{N})$, so from Proposition 3.11 we get $\ell^2(\mathbb{N}) = F_N \oplus F_N^{\perp}$.

2. (a) • Let $y = (y_n) \in G$, we show that $y \in F_N^{\perp}$; i.e., $y \perp x$, $\forall x \in F_N$. Indeed, for all $x = (x_n) \in F_N$, we have:

$$\langle x, y \rangle = \sum_{n=0}^{\infty} x_n y_n = \sum_{n=0}^{N} x_n y_n + \sum_{n=N+1}^{\infty} x_n y_n.$$

Since $x \in F_N$, then $\sum_{n=0}^{N} x_n = 0$, and since $y = (y_n) \in G$, we have $y_i = y_j = y^*$ for all $0 \leq i < j \leq N$ and $y_n = 0$ for $n > N$. This gives:

$$\langle x, y \rangle = y^* \sum_{n=0}^{N} x_n + \sum_{n=N+1}^{\infty} x_n \times 0 = 0$$

hence $y \in F_N^{\perp}$ and therefore $G \subset F_N^{\perp}$.

• Conversely, let $y = (y_n) \in F_N^{\perp}$, then for all $x = (x_n)_n \in F_N$, we have $\langle x, y \rangle = 0$; i.e.

$$\sum_{n=0}^{\infty} x_n y_n = 0. \tag{5.10}$$

In particular, if we take $x = (x_n)_n$ with $x_k = 1$ and $x_{k+1} = -1$ for all $0 \leq k \leq N-1$ and $x_k = 0$ for all $k \geq N+1$, we see that our choice is valid since this element belongs to F_N, thus from relation (5.10) we obtain:

$$\sum_{n=0}^{\infty} x_n y_n = 0 \Leftrightarrow x_k - x_{k+1} = 0 \Leftrightarrow x_k = x_{k+1}, \quad \forall 0 \leq k \leq N-1.$$

Moreover, if we take $x = (x_n)_n$ with $x_k = 0$ for all $0 \leq k \leq N-1$ and $x_k = y_k$ for all $k \geq N+1$, this element belongs to F_N, so from relation (5.10) we get:

$$\sum_{n=0}^{\infty} x_n y_n = 0 \Leftrightarrow y_k^2 = 0 \Leftrightarrow y_k = 0 \quad \forall k \geq N+1.$$

We deduce that $y \in G$, hence $F_N^{\perp} \subset G$. Thus, $F_N^{\perp} = G$.

(b) We compute the distance from a to F_N. We have

$$d(a, F_N) = \|a - P_{F_N}(a)\|,$$

where $P_{F_N}(a)$ is the orthogonal projection of a onto F_N. ($P_{F_N}(a)$ exists and is unique since $(\ell^2(\mathbb{N}), \langle \cdot, \cdot \rangle)$ is a Hilbert space and F_N is a closed subspace). Since $P_{F_N}(a) \in F_N$, let $P_{F_N}(a) = b = (b_n)_{n\in\mathbb{N}}$, then:

$$\sum_{n=0}^{N} b_n = 0, \tag{5.11}$$

First, compute $P_{F_N}(a) = b$. From the projection theorem:

$$a - b \in F_N^{\perp} = G,$$

so for all $0 \leq n \leq N$, we have $a_n - b_n = a_{n+1} - b_{n+1}$, hence $b_{n+1} - b_n = a_{n+1} - a_n = 0$, thus:

$$b_{n+1} = b_n, \quad \forall 1 \leq n \leq N. \tag{5.12}$$

For $n = 0$, we have

$$b_1 - b_0 = -1, \tag{5.13}$$

For all $n \geq N$, we have $a_n - b_n = 0$, hence $b_n = a_n = 0$. So it remains to compute b_0 and b_1 to determine $P_{F_N}(a)$. From (5.11) and (5.12), we have:

$$\sum_{n=0}^{N} b_n = 0 \Leftrightarrow b_0 + \sum_{n=1}^{N} b_1 = 0 \Leftrightarrow b_0 + N b_1 = 0,$$

hence $b_1 = b_n = \frac{-1}{N+1}$, for all $1 \leq n \leq N$ and $b_0 = \frac{N}{N+1}$. Thus:

$$\begin{aligned}
\|a - P_{F_N}(a)\|^2 &= \sum_{n=0}^{\infty} (a_n - b_n)^2 \\
&= (a_0 - b_0)^2 + \sum_{n=1}^{N} (a_n - b_n)^2 \\
&= (1 - \frac{N}{N+1})^2 + N\left(\frac{1}{N+1}\right)^2 \\
&= \frac{1}{(N+1)^2}.
\end{aligned}$$

Finally, $d(a, F_N) = \frac{1}{N+1}$.

Solution to Exercise 3.23

1. It is clear that $E = C([0,1], \mathbb{R})$ is a vector space. We verify that $\langle \cdot, \cdot \rangle$ is an inner product. Symmetry, bilinearity, and positivity are obvious. It remains to show that it is definite; for all $f \in E$, we have:

$$\begin{aligned}
\langle f, f \rangle = 0 \quad &\Leftrightarrow \quad f^2(0) + \int_0^1 f'^2(t)\, dt = 0 \\
&\Leftrightarrow \quad f^2(0) = 0 \quad \text{and} \quad f'^2(t) = 0, \quad \forall t \in]0,1[\\
&\Leftrightarrow \quad f(0) = 0 \quad \text{and} \quad f(t) = c, \quad \forall t \in]0,1[\\
&\Leftrightarrow \quad f(t) = 0, \quad \forall t \in [0,1] \Leftrightarrow f = 0.
\end{aligned}$$

2.

$$\begin{aligned}
\Big(Vect(e_0)\Big)^{\perp} &= \{f \in E : \langle f, e_0 \rangle = 0\} \\
&= \{f \in E : f(0)e_0(0) + \int_0^1 f'(t)e_0'(t)\, dt = 0\} \\
&= \{f \in E : f(0) = 0\}.
\end{aligned}$$

3. We note that $G = \{f \in E : f(0) = 0\} = \Big(Vect(e_0)\Big)^{\perp}$, hence $G^{\perp} = \Big(Vect(e_0)\Big)^{\perp\perp}$. Since $Vect(e_0)$ is a closed subspace, it follows that $G^{\perp} = \Big(Vect(e_0)\Big)^{\perp\perp} = Vect(e_0)$.

Solution to Exercise 3.24

1. The application $T : E \to \mathbb{R}$ defined by $T(f) = f(0)$ is linear, hence $H = \ker T$, so H is a hyperplane in E. Clearly, $G = Vect\{1\}$ where 1 is the constant function equal to 1.

2. Since $1(0) = 1 \neq 0$, we have $1 \notin H$, so $G \nsubseteq H$. As H is a hyperplane, we have $E = H \oplus G$. (As a second method, one can give a direct proof).

3. By definition:

$$\begin{aligned}
H^{\perp} &= \{f \in E : \int_0^1 f(t)h(t)\, dt = 0, \forall h \in H\} \\
G^{\perp} &= \{g \in E : \int_0^1 f(t)g(t)\, dt = 0, \forall g \in G\} \\
&= \{g \in E : \int_0^1 f(t)\, dt = 0\} \quad \text{(since functions in } G \text{ are constant).}
\end{aligned}$$

4. Let $g \in H^{\perp}$:

   (a) We have

$$\int_0^1 g(t)h(t)\, dt = 0, \forall h \in H \tag{5.14}$$

   Taking $h(t) = tg(t)$, we have $h \in E$ and $h(0) = 0$ so $h \in H$, and thus the function $t \mapsto tg(t)$ satisfies relation (5.14); i.e.,

$$\int_0^1 tg(t)^2\, dt = 0. \tag{5.15}$$

(b) Since the function $[0,1] \ni t \mapsto tg^2(t)$ is continuous and non-negative, from relation (5.15), we have: $tg^2(t) = 0$ for all $t \in [0,1]$. As $t \mapsto t$ is strictly positive on $]0,1]$, it follows that $g(t) = 0$ for all $t \in]0,1]$. But g is continuous at 0, so $0 = \lim_{t\to 0} g(t) = g(0)$, hence $g(0) = 0$. We conclude that $g(t) = 0$ for all $t \in [0,1]$.

(c) We deduce that for all $g \in H^\perp$, we have $g = 0$, which means $H^\perp = \{0\}$.

5. We have:

$$\begin{aligned} H^{\perp\perp} &= \{0\}^\perp = E \neq H \\ H \oplus H^\perp &= H \oplus \{0\} = H \neq E. \\ H^\perp + G^\perp &= \{0\} + G^\perp = G^\perp \neq (H \cap G)^\perp = \{0\}^\perp = E. \end{aligned}$$

These results contradict Propositions 3.11 and 3.12, because in this exercise the space $(E, \langle\cdot,\cdot\rangle)$ is a pre-Hilbert space and not a Hilbert space.

Solution to Exercise 3.30: See the correction of the L3-Maths 2019-2020 exam (Chapter 6).

Solution to Exercise 3.31: See the correction of the L3-Maths 2017-2018 resit exam (Chapter 6).

Solution to Exercise 3.32: See the correction of the L3-Maths 2017-2018 final exam (Chapter 6).

Solution to Exercise 3.34: See the correction of the L3-Maths 2017-2018 final exam (Chapter 6).

Solution to Exercise 3.35: See the correction of the L3-Maths 2018-2019 final exam (Chapter 6).

Correction of Exercise 3.38: We solve only the third question.

We define:

$$h(x) = \begin{cases} x^2, & \text{if } 0 \le x \le a; \\ 0, & \text{if } a < x \le 1. \end{cases}$$

Suppose there exists $g \in E$ such that $\varphi_a(f) = \langle g, f\rangle$ for all $f \in E$; that is,

$$\int_0^a t^2 f(t)\, dt = \int_0^1 g(t) f(t)\, dt$$

Since $\int_0^a t^2 f(t)\, dt = \int_0^1 h(t) f(t)\, dt$, then

$$\int_0^1 h(t) f(t)\, dt = \int_0^1 g(t) f(t)\, dt, \quad \forall f \in E$$

The inner product of E extends to the space $L^2([0,1])$, so we deduce that $\langle h, f\rangle = \langle g, f\rangle$, i.e., $\langle h - g, f\rangle = 0$ for all $f \in E$. Since E is dense in $L^2([0,1])$, then

$$\langle h - g, f\rangle = 0, \quad \forall f \in L^2([0,1]).$$

Setting $f = h - g$, we find $h = g$, which is absurd since $h \notin E$.

The conclusion is that the Riesz representation theorem is false in a pre-Hilbert space.

Correction of Exercise 3.40:

1. Let $x \in H$, we have:

$$\begin{aligned} \|S_n(x)\|^2 &= \langle \sum_{k=0}^{n} \alpha_i \langle x, e_i \rangle f_i, \sum_{k=0}^{n} \alpha_k \langle x, e_k \rangle f_k \rangle \\ &= \sum_{i=0}^{n} \sum_{k=0}^{n} \alpha_i \alpha_k \langle x, e_i \rangle \langle x, e_k \rangle \langle f_i, f_k \rangle, \end{aligned}$$

Since the family $\{f_n\}$ is orthogonal, then $\langle f_i, f_k \rangle = \begin{cases} 1, & \text{if } i = k; \\ 0, & \text{if } i \neq k. \end{cases}$

2. It is clear that for all $n \in \mathbb{N}$, the map $S_n : H \ni x \mapsto S_n(x)$ is linear. We show that it is continuous. Let $M = \sup_{n \in \mathbb{N}} |\alpha_n|$. We have

$$\|S_n(x)\|^2 = \sum_{k=0}^{n} \alpha_k^2 |\langle x, e_k \rangle|^2.$$

Thanks to Bessel's inequality, we obtain:

$$\|S_n(x)\|^2 \leq M^2 \sum_{k=0}^{n} |\langle x, e_k \rangle|^2 \leq M^2 \|x\|^2,$$

hence: $\|S_n(x)\| \leq M\|x\|$.

3. We obtain the linearity of u by taking the limit in the expression

$$S_n(\lambda x + \mu y) = \lambda S_n(x) + \mu S_n(y), \quad \forall x, y \in H, \ \forall \lambda, \mu \in \mathbb{R}.$$

Previously, we had:

$$\|S_n(x)\| \leq M\|x\|, \quad \forall n \in \mathbb{N}, \ \forall x \in H.$$

Taking the limit, we get:

$$\|u(x)\| \leq M\|x\|, \quad \forall x \in H.$$

which shows that u is continuous.

4. We have:

$$\begin{aligned} x \in \ker u \Leftrightarrow u(x) = 0 &\Leftrightarrow \|u(x)\| = 0 \\ &\Leftrightarrow \alpha_n^2 |\langle x, e_n \rangle|^2 = 0, \quad \forall n \in \mathbb{N} \\ &\Leftrightarrow \langle x, e_n \rangle = 0, \quad \forall n \in \mathbb{N} \\ &\Leftrightarrow x \in \{e_n\}^{\perp} \end{aligned}$$

hence $\ker u = \{e_n\}^{\perp}$. It follows that:

$$\begin{aligned} u \text{ is injective } \Leftrightarrow \ker u = \{0\} &\Leftrightarrow \{e_n\}^{\perp} = \{0\} \\ &\Leftrightarrow \overline{\{e_n\}} = H. \end{aligned}$$

Moreover, since $\{e_n\}$ is an orthonormal family, it forms a Hilbert basis of H.