We conclude that the statements $(i), (ii), (iii)$ and (iv) are equivalent. ■

Remark 3.13 *1. If E is a complex pre-Hilbert vector space, the previous theorem remains valid by replacing assertion (iv) with:*

$$\langle x, y\rangle = \sum_{n=0}^{\infty} \overline{\langle x, e_n\rangle}\langle y, e_n\rangle, \quad \forall x, y \in E,$$

where $\overline{\langle x, e_n\rangle}$ is the complex conjugate of $\langle x, e_n\rangle$.

2. An orthonormal family is a Hilbert basis of a pre-Hilbert space if and only if it satisfies one of the assertions (ii), (iii), or (iv).

Corollary 3.7 *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, F a closed vector subspace of H, and $\{e_n\}_{n\in\mathbb{N}}$ a Hilbert basis of F. For every $x \in H$, let $P_F(x)$ be the orthogonal projection of x onto F. Then the series with general term $\langle x, e_n\rangle e_n$ is convergent, and we have:*

$$P_F(x) = \sum_{n=0}^{\infty} \langle x, e_n\rangle e_n.$$

Proof. The projection is characterized by:

$$\langle x - P_F(x), e_n\rangle = 0, \quad \forall n \in \mathbb{N},$$

therefore $\langle x, e_n\rangle = \langle P_F(x), e_n\rangle$, for all $n \in \mathbb{N}$. From Theorem 3.8 and by applying the identity 3.18 to the element $P_F(x)$ of the Hilbert space F, we obtain:

$$P_F(x) = \sum_{n=0}^{\infty} \langle P_F(x), e_n\rangle e_n = \sum_{n=0}^{\infty} \langle x, e_n\rangle e_n.$$

■

3.10 Exercises

♦ Generalities on Pre-Hilbert Spaces

Exercice 3.9 *Show that the application $\langle\cdot,\cdot\rangle$ is an inner product on the vector space E in the following cases:*

1. $\langle P, Q\rangle = \int_0^1 P(x)Q(x)\,dx \qquad \forall P, Q \in E = \mathbb{R}[x]$*, the space of polynomials.*
2. $\langle A, B\rangle = tr(A^T B) \qquad \forall A, B \in E = \mathcal{M}_n(\mathbb{R})$*, the space of square matrices with n rows and n columns.*

Exercice 3.10 *Let E be a vector space equipped with an inner product denoted by $\langle\cdot,\cdot\rangle$ and associated norm $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$. Show the polarization identity:*

$$\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2).$$

Exercice 3.11 *Show that the inner product defines a continuous mapping from* $E \times E$ *to* $\mathbb{R}$.

Exercice 3.12 *Let* E *be the set of functions in* $C^\infty([-1,1],\mathbb{R})$ *such that for every* $n \in \mathbb{N}$, *we have:*

$$f^{(n)}(-1) = f^{(n)}(1) = 0.$$

1. *Show that the application* $\langle \cdot,\cdot \rangle : E \times E \to \mathbb{R}$ *defined for all* $f, g \in E$ *by*

$$\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)\, dt$$

   is an inner product on E.

2. *Show that* $D : E \ni f \mapsto f'$ *is an endomorphism of* E *satisfying, for all* $f, g \in E$,

$$\langle D(f), g \rangle = -\langle f, D(g) \rangle.$$

Exercice 3.13 *Let* E *be a pre-Hilbert space with inner product* $\langle \cdot,\cdot \rangle$ *and associated norm* $\|\cdot\| = \sqrt{\langle \cdot,\cdot \rangle}$. *Let* $u, v \in E$, *and define:*

$$w = u - \frac{\langle u, v \rangle}{\|v\|^2} v, \qquad (v \neq 0).$$

1. *Show that* $\|w\|^2 = \|u\|^2 - \dfrac{|\langle u, v \rangle|^2}{\|v\|^2}$.

2. *Deduce the Cauchy-Schwarz inequality.*

3. *Write explicitly the Cauchy-Schwarz inequality in the following cases:*

   (a) $E = \mathbb{R}^N$, $\langle X, Y \rangle = \sum_{i=1}^{N} x_i y_i$; $\quad X = (x_1, x_2, ..., x_N), Y = (y_1, y_2, ..., y_N) \in \mathbb{R}^N$.

   (b) $E = C([-1,1],\mathbb{R})$, $\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)\, dt$; $\quad f, g \in C([-1,1],\mathbb{R})$.

Exercice 3.14 *Let* $(E, \langle \cdot,\cdot \rangle)$ *be a pre-Hilbert space and let* $x, y \in E$. *The goal of this exercise is to prove that the following conditions are equivalent:*

(i) $\|x + y\| = \|x\| + \|y\|$,

(ii) $\langle x, y \rangle = \|x\|\|y\|$,

(iii) *There exist* $a, b \geq 0$ *such that* $ax = by$.

1. *Show that (i)* $\Rightarrow$ *(ii).*

2. (a) *For all* $y \neq 0$, *compute* $\left\| x - \dfrac{\langle x, y \rangle}{\|y\|^2} y \right\|^2$.

   (b) *Deduce that (ii)* $\Rightarrow$ *(iii).*

3. *Finally, verify that (iii)* $\Rightarrow$ *(i) and conclude.*

Exercice 3.15 *Let $H = C([0, \frac{\pi}{2}], \mathbb{R})$, the vector space of real-valued continuous functions on $[0, \frac{\pi}{2}]$, equipped with the norm*

$$\|f\|_\infty = \sup_{x \in [0, \frac{\pi}{2}]} |f(x)|, \quad f \in H.$$

Using the functions $f(t) = \cos t$ and $g(t) = \sin t$, show that H is not a pre-Hilbert space.

Exercice 3.16 *Let $E = C([-1, 1], \mathbb{R})$, the space of continuous functions on $[-1, 1]$, equipped with the following inner product:*

$$\langle u, v \rangle = \int_{-1}^{1} u(t)v(t)\, dt, \quad \forall u, v \in E.$$

Is the space E a Hilbert space? (See Exercise 1.9.)

Exercice 3.17 *Let $H = C^1([0, 1], \mathbb{R})$, the space of continuously differentiable functions. We define the application:*

$$F : H \times H \to \mathbb{R}, \quad F(u, v) = \int_0^1 (u(t)v(t) + u'(t)v'(t))\, dt.$$

1. *Show that F is an inner product on H.*
2. *Consider the sequence (u_n) defined by:*

$$u_n(x) = \begin{cases} nx, & \text{if } x \in \left[0, \frac{1}{n^4}\right], \\ \frac{4}{3}x \cdot \frac{3}{4} - \frac{1}{3n^2}, & \text{if } x \in \left[\frac{1}{n^4}, 1\right]. \end{cases}$$

   (a) *Show that $u_n \in H$, for all $n \in \mathbb{N}^*$.*

   (b) *Show that (u_n) is a Cauchy sequence.*

   (c) *Is the space H a Hilbert space?*

♦ On Orthogonality

Exercice 3.18 *Let $E = C([0, 1], \mathbb{C})$ be the space of continuous functions on $[0, 1]$ with values in $\mathbb{C}$, equipped with the inner product:*

$$\langle f, g \rangle = \int_0^1 f(t)\overline{g(t)}\, dt,$$

verify that the family $(f_n)_{n \in \mathbb{Z}}$ in E defined by $f_n(x) = e^{2i\pi nx}$, $x \in [0, 1]$, $n \in \mathbb{Z}$ is orthonormal.

Exercice 3.19 *We consider the space $H = C([-1, 1], \mathbb{R})$. We define the following application:*

$$\langle f, g \rangle = \frac{2}{\pi} \int_{-1}^{1} \frac{1}{\sqrt{1 - x^2}} f(x)g(x)\, dx, \tag{3.22}$$

1. *Verify that the integral in (3.22) is well-defined, then show that $\langle \cdot, \cdot \rangle$ is an inner product on E.*

2. *We consider the family $(T_n)_{n\in\mathbb{N}}$ defined by:*

$$T_n(x) = \cos(n \arccos x), \quad x \in [-1,1] \tag{3.23}$$

*Verify that this family is orthogonal and orthonormal for $n \in \mathbb{N}^*$. (The family in (3.23) is known as the "**Chebyshev polynomials**").*

Exercice 3.20 1. *Let E be a pre-Hilbert space and let $F = Vect\{e_1, e_2, ..., e_N\}$ be the subspace generated by $e_1, e_2, ..., e_N \in E$. Show that for all $x \in E$, we have:*

$$x \perp F \Leftrightarrow x \perp e_i \quad \forall i = 1, 2, ..., N.$$

2. *Let $E = \mathbb{R}^4$ equipped with the canonical inner product and let F be the subspace defined by*

$$F = \{X = (x,y,z,t) \in \mathbb{R}^4 : x+y+z+2t = 0,\ x+t = 0\}$$

Determine $F^\perp$, the orthogonal complement of F.

Exercice 3.21 *Let $H = l^2(\mathbb{N}) = \{x = (x_n)_{n\in\mathbb{N}}; \sum\limits_{n=0}^{\infty} x_n^2 < \infty\}$ equipped with its usual inner product defined by:*

$$\langle x, y\rangle = \sum_{n=0}^{\infty} x_n y_n; \quad \forall x = (x_n)_{n\in\mathbb{N}},\ y = (y_n)_{n\in\mathbb{N}} \in l^2(\mathbb{N}).$$

Compute the orthogonal complement of the following set:

$$M = \{x = (x_n) \in H : \quad x_{2n} = 0, \quad n \in \mathbb{N}\}.$$

Exercice 3.22 *Let $l^2(\mathbb{N})$ be equipped with the usual inner product. For every $N \in \mathbb{N}$, denote by F_N the subspace of $l^2(\mathbb{N})$ defined by*

$$F_N = \{x = (x_n) \in \ell^2(\mathbb{N}) : \quad \sum_{n=0}^{N} x_n = 0\}.$$

1. (a) *Show that the application*

$$\begin{aligned} J : \ell^2(\mathbb{N}) &\to \mathbb{R} \\ x = (x_n) &\mapsto J(x) = \sum_{n=0}^{N} x_n \end{aligned}$$

is linear and continuous.

   (b) *What can be deduced about F_N? Conclude that $\ell^2(\mathbb{N}) = F_N \oplus F_N^\perp$.*

2. *Let*

$$G = \Big\{ y = (y_n) \in \ell^2(\mathbb{N}) \ : \ y_i = y_j \text{ for all } 0 \le i < j \le N, \text{ and } y_n = 0 \text{ for } n > N \Big\}$$

   (a) *Show that $F_N^\perp = G$.*

(b) *Let $a = (a_n)_{\in \mathbb{N}}$ such that $a_0 = 1$ and $a_n = 1$ for $n \geq 1$. Compute the distance from a to F_N.*

Exercice 3.23 *Let $E = C([0,1], \mathbb{R})$. For all $f, g \in E$ define:*

$$\langle f, g \rangle = f(0)g(0) + \int_0^1 f'(t)g'(t)\, dt$$

1. *Show that $(E, \langle \cdot, \cdot \rangle)$ is a pre-Hilbert space.*
2. *Let the function e_0 in E be defined by $e_0(t) = 1$. Determine $\Big(Vect(e_0)\Big)^{\perp}$.*
3. *Let $G = \{f \in E : \quad f(0) = 0\}$. Determine $G^{\perp}$.*

Exercice 3.24 *Let $E = C([0,1], \mathbb{R})$ be equipped with the inner product $\langle f, g \rangle = \int_0^1 f(t)g(t)\, dt$. Let $H = \{f \in E : f(0) = 0\}$ and G be the set of constant functions on $[0,1]$ taking values in $\mathbb{R}$.*

1. *Justify that H is a hyperplane in E and G is a one-dimensional vector subspace of E.*
2. *Deduce that $E = H \oplus G$.*
3. *Define $H^{\perp}$ and $G^{\perp}$.*
4. *Let $g \in H^{\perp}$.*
   (a) *Prove that $\int_0^1 tg^2(t)\, dt = 0$.*
   (b) *Deduce that $g(t) = 0$ for all $t \in [0,1]$.*
   (c) *Determine $H^{\perp}$.*
5. *Do we have: $H^{\perp\perp}$; $H \oplus H^{\perp} = E$; $H^{\perp} + G^{\perp} = (H \cap G)^{\perp}$?*

Exercice 3.25 *Let the vector space $E = C([0,1])$ be equipped with the inner product*

$$\langle f, g \rangle = \int_0^1 f(t)\overline{g(t)}\, dt$$

Let F be the vector subspace defined by

$$F = \Big\{f \in C([0,1]); f(x) = 0 \text{ for all } x \in \Big[0, 1/2\Big]\Big\}$$

1. *Show that F is a closed subspace.*
2. *Show that $F^{\perp}$ is the vector subspace of functions vanishing on $\Big[1/2, 1\Big]$.*
3. *Deduce that for a closed subspace in a pre-Hilbert space, its orthogonal is not necessarily a complement.*

Exercice 3.26 *Let $E = C([-1,1])$ be the space of continuous functions on $[-1,1]$ with complex values, equipped with the Hermitian inner product*

$$\langle f, g \rangle = \int_{-1}^1 f(t)\overline{g(t)}\, dt.$$

We consider the subspace P defined by

$$P = \{f \in E : \quad f(-t) = f(t), \quad \forall t \in [-1,1]\}.$$

1. *Compute* $P^\perp$ *and* $(P^\perp)^\perp$.

2. *Show that* $E = P \oplus P^\perp$.

3. *Show without computation that* P *is closed in* E.

Exercice 3.27 *Let* $(E, \langle\cdot,\cdot\rangle)$ *be a real pre-Hilbert space. Let* F *and* G *be two vector subspaces of* E *satisfying* $F \subset G^\perp$ *and* $F + G = E$. *Show that* $F = G^\perp$ *and then that* $F^\perp = G$.

♦ On Orthogonal Projection

Exercice 3.28 *Let* H *be a Hilbert space and* F *a non-empty subset of* H.

1. *Let* $x \in H$. *Write a characterization of the orthogonal projection of* x *onto* F *(denoted* $P_F(x)$*) in the case where* F *is a closed convex set, and in the case where* F *is a closed subspace.*

2. *Let* $\overline{B}(0,1)$ *be the closed unit ball in* H.

   (a) *Verify that* $\overline{B}(0,1)$ *is convex.*

   (b) *Verify that for all* $x \in H$, $x \neq 0$, *and* $y \in H$:

$$\left\langle x - \frac{1}{\|x\|}x, y - \frac{1}{\|x\|}x \right\rangle = (\|x\| - 1)\left(\frac{1}{\|x\|}\langle x, y\rangle - 1\right).$$

   (c) *Show that for all* $x \in H$,

$$P_F(x) = \begin{cases} x, & \text{if } x \in \overline{B}(0,1); \\ \frac{1}{\|x\|}x, & \text{if } x \notin \overline{B}(0,1). \end{cases}$$

Exercice 3.29 *Let* $E = C([0,1], \mathbb{R})$ *equipped with the norm:*

$$\|f\| = \|f\|_\infty + \int_0^1 |f(t)|\, dt, \qquad f \in E.$$

Let $F = \{f \in E : \quad f(0) = 0\}$. *We denote by* $\mathbf{1}$ *the constant function equal to 1.*

1. *Verify that* F *is a convex, closed subset of* E.

2. *Compute* $\mathrm{dist}(1, F)$. *Does* $\mathbf{1}$ *admit a projection onto* F*?*

Exercice 3.30 *Let* E *be the vector space of real polynomial functions defined on* $\mathbb{R}^+$ *of degree at most 3, equipped with the inner product*

$$\langle f, g\rangle = \int_0^{+\infty} f(t)g(t)e^{-t}\, dt.$$

Let F *be the vector subspace of* E *consisting of real polynomial functions on* $\mathbb{R}_+$ *of degree at most 2. Let* $q_i(X) = X^i$ *for* $i = 0, 1, 2, 3$.

1. *Verify that for all* $n \in \mathbb{N}$, $\int_0^{+\infty} t^n e^{-t}\, dt = n!$.

2. *Determine an orthonormal basis of F.*

3. *Compute $P_F(X^n)$ and $d(X^n, F)$ for all $n \in \mathbb{N}$.*

Exercice 3.31 *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $a \neq 0_H$ an element of H. Let $\{a\}^\perp$ denote the orthogonal of the set $\{a\}$.*

1. *Show that $\{a\}^\perp$ is a closed subspace.*

2. (a) *Let $x \in H$ and $P_F(x)$ be the orthogonal projection of x onto $F = \{a\}^\perp$. Using the characterization of $P_F(x)$, show that there exists $\lambda \in \mathbb{R}$ such that $x = \lambda a + P_F(x)$.*
   (b) *Verify that: $|\langle x, a\rangle| = |\lambda| \|a\|^2$.*
   (c) *Deduce that $d(x, \{a\}^\perp) = \frac{|\langle x,a\rangle|}{\|a\|}$.*

Exercice 3.32 *Let $E = \mathbb{R}[X]$ be the space of real-coefficient polynomials equipped with the inner product: $\forall P, Q \in E : \quad \langle P, Q\rangle = \int_{-1}^{1} P(x)Q(x)\, dx$.*

1. *Let $F = Vect\{1, X, X^2\}$ be the space of real polynomials of degree ≤ 2. Determine an orthonormal basis $\{P_0, P_1, P_2\}$ of F_2.*

2. *For all $f \in E$, denote by $P_F(f)$ the orthogonal projection of f onto F.*
   (a) *Justify the existence and uniqueness of $P_F(f)$.*
   (b) *Show that $P_F(f) = \sum_{i=0}^{2} \langle f, P_i\rangle P_i$.*
   (c) *Let $g(X) = X^3$, compute $P_F(g)$.*
   (d) *Compute the distance from g to F and deduce the value of α such that*

$$\alpha = \min_{a,b,c \in \mathbb{R}} \int_{-1}^{1} |X^3 - aX^2 - bX - c|^2\, dx.$$

Exercice 3.33 *Let the Hilbert space*

$$H = \{f : [1, +\infty[\to \mathbb{R} : \quad \int_1^{+\infty} f^2(x) \frac{1}{x^6}\, dx < \infty\},$$

equipped with the inner product $\langle f, g\rangle = \int_1^{+\infty} f(x)g(x) \frac{1}{x^6}\, dx$

1. (a) *Determine the integers $n \in \mathbb{N}$ for which the function $e_n(x) = x^n$ belongs to H.*
   (b) *Deduce that $\mathbb{R}_2[x]$, the space of polynomials of degree ≤ 2, is contained in H.*

2. *Compute $\langle e_k, e_l\rangle$ for $k, l \in \{0, 1, 2\}$.*

3. *Determine $a, b \in \mathbb{R}$ such that $\{P_0, P_1, P_2\}$ is an orthogonal system with $P_0(x) = 1$, $P_1(x) = 4x + a$, $P_2(x) = 3x^2 + bx + 10$*

4. (a) *Let $f(x) = \frac{1}{x}$. Verify that $f \in H$.*
   (b) *Determine the projection of f onto $\mathbb{R}_2[x]$.*

Exercice 3.34 *Let $H = L^2([0,1])$ be the space of functions $f : [0,1] \to \mathbb{R}$ such that*

$$\int_0^1 f(x)^2 \, dx < \infty.$$

We assume that $L^2([0,1])$ is a Hilbert space when equipped with the inner product

$$\forall f, g \in H : \quad \langle f, g \rangle = \int_0^1 f(x)g(x) \, dx.$$

1. *Let T be the application defined by*

$$\begin{array}{rcl} T : H & \to & \mathbb{R} \\ f & \mapsto & T(f) = \displaystyle\int_0^1 f(t) \, dt \end{array}$$

   (a) *Show that T is a linear and continuous map.*

   (b) *Find a function $g \in H$ such that $T(f) = \langle f, g \rangle$, (note that such g exists and is unique; justify this).*

2. *Let F be a subset of H defined by: $F = \{f \in H : \quad \int_0^1 f(t) \, dt = 0\}$*

   (a) *Show that F is a closed subspace.*

   (b) *Verify that $F = \{g\}^\perp$.*

   (c) *Determine the projection of the function e^t onto F.*

Exercice 3.35 *Let $H = L^2([-1,1]) = \{f : [-1,1] \to \mathbb{R}$ measurable s.t. $\int_{-1}^1 |f(x)|^2 \, dx < \infty\}$ equipped with the inner product*

$$\forall f, g \in H : \langle f, g \rangle = \int_{-1}^1 f(x)g(x) \, dx.$$

Let T be the application defined on H by:

$$\begin{array}{rcl} T : H & \to & \mathbb{R} \\ f & \mapsto & T(f) = \displaystyle\int_{-1}^0 f(x) \, dx - \int_0^1 f(x) \, dx. \end{array}$$

1. *Show that T is linear and continuous.*

2. *Let $g : [-1,1] \to \mathbb{R}$ be defined by:*

$$g(x) = \begin{cases} 1, & \text{if } -1 \leq x < 0 \\ -1, & \text{if } 0 \leq x \leq 1 \end{cases}$$

   (a) *Verify that $g \in H$.*

   (b) *Show that $\{g\}^\perp = \ker T$.*

   (c) *Let $h(x) = e^{-x}$, compute the projection of h onto $\ker T$.*

Exercice 3.36 *Let us consider $E = C([0,1], \mathbb{R})$ equipped with the norm $\| \cdot \|_\infty$. Recall that $(E, \| \cdot \|_\infty)$ is complete and that the norm $\| \cdot \|_\infty$ is not associated with any inner product on E (see Exercise 3 from Series 2). We consider the set:*

$$F = \{f \in E : f(0) = 0 \text{ and } \int_0^1 f(x) \, dx \geq 1\}$$

1. *Show that F is closed.*

2. *Show that F is convex.*

3. *Show that for every $f \in F$, we have $\|f\|_\infty > 1$.*

4. *Show that* $\inf\limits_{f \in F} \|f\|_\infty = 1$

5. *Deduce that $d(0_E, F)$ is not attained. What can we conclude?*

⧫ About Riesz's Theorem

Exercice 3.37 *Let E be the pre-Hilbert space defined by:*

$$E = \{(x_n)_{n\in\mathbb{N}}; \exists n_0 \in \mathbb{N}, \forall n \geq n_0 : x_n = 0\},$$

equipped with the inner product $\langle x, y\rangle = \sum x_n y_n, \quad x = (x_n)_{n\in\mathbb{N}}, y = (y_n)_{n\in\mathbb{N}}$.

1. *Show that the map $\varphi : E \to \mathbb{R}$ defined by* $\varphi(x) = \sum\limits_{n\geq 1} \dfrac{x_n}{n}$, $(x = (x_n)_{n\in\mathbb{N}})$ *is a continuous linear form on E.*

2. *Does there exist an element $a = (u_n) \in E$ such that for all $x \in E$ we have: $\varphi(x) = \langle a, x\rangle$?*

3. *What can we conclude about E?*

Exercice 3.38 *Let $E = C([0,1])$, the space of continuous functions on $[0,1]$ with real values, equipped with the usual inner product:*

$$\langle f, g\rangle = \int_0^1 f(t)g(t)\, dt.$$

We consider the map $\varphi_a : E \to \mathbb{R}$, with $a \in]0,1[$, defined by

$$\varphi_a(f) = \int_0^a t^2 f(t)\, dt$$

1. *Show that E is not complete. Hint: consider the sequence f_n defined by*

$$f_n(x) = \begin{cases} n, & \text{if } x \in [0, \frac{1}{n^3}] \\ x^{-\frac{1}{3}}, & \text{if } x \in [\frac{1}{n^3}, 1]. \end{cases}$$

2. *Show that u is a continuous linear form on E. Compute its norm.*

3. *Show that there does not exist a vector $g \in E$ such that $\varphi_a(f) = \langle f, g\rangle$ for all $f \in E$. What can we conclude?*

Exercice 3.39 (Lax-Milgram Theorem) *Let H be a Hilbert space, $L(\cdot) : H \ni v \mapsto L(v) \in \mathbb{R}$ a continuous linear form on H, and $a(\cdot,\cdot)$ a continuous bilinear form with*

$$\begin{aligned} a(\cdot,\cdot) : H \times H &\to \mathbb{R} \\ (u, v) &\mapsto a(u, v). \end{aligned}$$

Suppose that $a(\cdot,\cdot)$ *is coercive on* H*, i.e., there exists* $\alpha > 0$ *such that*

$$a(v, v) \geq \alpha\|v\|^2.$$

We consider the problem:

$$\begin{cases} \text{Find } u \in H; \\ a(u, v) = L(v), \quad \text{for all } v \in H. \end{cases} \tag{3.24}$$

We want to prove that problem (3.24) admits a unique solution.

1. (a) *Show that* $T_w : H \ni v \mapsto T_w(v) = a(w, v)$ *is linear and continuous.*

   (b) *Deduce that there exists an element denoted* $A(w) \in H$ *such that for all* $v \in H$*, we have*

   $$a(w, v) = \langle A(w), v\rangle$$

   (c) *Show that the operator* $A : H \ni w \mapsto A(w) \in H$ *is linear and verify that*

   $$\|A(w)\|^2 = a(w, A(w)) \leq M\|w\|\|A(w)\|$$

   (d) *Deduce that the operator* $w \mapsto A(w)$ *is continuous.*

2. (a) *Justify the existence of an element* $f \in H$ *such that for all* $v \in H$*, we have*

   $$L(v) = \langle f, v\rangle.$$

   (b) *Verify that problem (3.24) is equivalent to:*

   $$\text{Find } u \in H \text{ such that } A(u) = f.$$

3. (a) *Using the coercivity of* $a(w, v)$*, show that for all* $w \in H$

   $$\alpha\|w\| \leq \|A(w)\|. \tag{3.25}$$

   (b) *Deduce that* A *is injective.*

4. (a) *Let* $(A(w_n))$ *be a sequence in Im*(A) *such that* $A(w_n)$ *converges to* b *in* H*. By virtue of relation (3.25), show that* $(A(w_n))$ *is a Cauchy sequence and deduce that the sequence* (w_n) *converges to* $w \in H$*.*

   (b) *Deduce that* $b \in$ *Im*(A) *and consequently Im*(A) *is closed.*

   (c) *Using the coercivity of* $a(w, v)$*, verify that* $(Im(A))^\perp = \{0\}$*.*

   (d) *Deduce that* A *is surjective.*

5. *Conclusion:* A *is bijective and therefore the equation* $A(u) = f$ *admits a unique solution, which completes the proof.*

⧫ On Hilbert Bases

Exercice 3.40 *Let H be a Hilbert space, and let $(e_n)_n$ and $(f_n)_n$ be two orthonormal families in H. Let $(\alpha_n)_n$ be a bounded, non-zero real sequence. Suppose that for all $x \in H$, the series $\sum \alpha_n \langle x, e_n\rangle f_n$ converges. We define $S_n(x) = \sum_{k=0}^{n} \alpha_k \langle x, e_k\rangle f_k$*

1. *Verify that $\|S_n(x)\|^2 = \sum_{k=0}^{n} \alpha_k^2 |\langle x, e_k\rangle|^2$.*

2. *Show that S_n is linear and continuous.*

3. *For all $x \in E$, define $u(x) = \sum_{n=0}^{\infty} \alpha_n \langle x, e_n\rangle f_n$. Deduce that u is linear and continuous.*

4. *Show that (e_n) is a Hilbert basis if and only if u is injective, more precisely:*

$$Ker u = \{0_H\} \Leftrightarrow \{e_n\}^{\perp} = \{0_H\}.$$

Exercice 3.41 (Haar Functions) *Consider the family of functions $(H_p)_{p\in\mathbb{N}}$ defined on $[0,1]$ by $H_0 = 1$ and, for $n \in \mathbb{N}$ and $1 \leq k \leq 2^n$,*

$$H_{2^n+k-1}(x) = \begin{cases} \sqrt{2^n}, & \text{if } x \in](2k-2)2^{-n-1}, (2k-1)2^{-n-1}[; \\ -\sqrt{2^n}, & \text{if } x \in](2k-1)2^{-n-1}, (2k)2^{-n-1}[; \\ 0, & \text{otherwise.} \end{cases}$$

1. *Plot H_0, H_1, and H_2. Show that*

$$H_{2^n+k-1}(x) = H_1(2^n x - k + 1)2^{\frac{n}{2}}.$$

2. *Prove that $(H_p)_{p\in\mathbb{N}}$ is an orthonormal system in $L^2([0,1])$. Show the following successively:*

   (a) *Show that for every n, the functions H_{2^n+k-1} are orthogonal among themselves for $1 \leq k \leq 2^n$.*

   (b) *Let $m > n$. Show that there is no integer ℓ such that*

   $$(2k-2)2^{-m-1} < \ell 2^{-n-1} < 2k2^{-m-1}.$$

   Deduce that for any integer ℓ such that $1 \leq \ell \leq 2^n$, the function $H_{2^n+\ell-1}$ is constant on the interval $](2k-1)2^{-m-1}, (2k)2^{-m-1}[$.

   (c) *Conclude.*

3. *Let $f \in L^2([0,1])$ such that for every $p \in \mathbb{N}$, $\int_0^1 f(x)H_p(x)\,dx = 0$. Define:*

$$F(y) = \int_0^y f(x)\,dx.$$

   (a) *Prove that for every $n \in \mathbb{N}$ and for all integers k such that $1 \leq k \leq 2^n$:*

   $$-F\Big(\frac{2k-2}{2^{n+1}}\Big) + 2F\Big(\frac{2k-1}{2^{n+1}}\Big) - F\Big(\frac{2k}{2^{n+1}}\Big).$$

(b) *Deduce that $F = 0$. (Note that F is continuous). You may use induction to show that $F(\frac{k}{2^n}) = 0$ for all odd integers k.*

(c) *Deduce that $f = 0$ and that $(H_p)_{p\in\mathbb{N}}$ is a Hilbert basis of $L^2([0,1])$.*

Exercice 3.42 (Legendre Polynomials) *Let $H = L^2([-1,1])$ be the Lebesgue space with the usual inner product denoted $\langle\cdot,\cdot\rangle$.*

1. *Prove by induction the following identity:*

$$\langle f^{(n)}, g\rangle = (-1)^n\langle f, g^{(n)}\rangle + \Big[\sum_{m=0}^{n-1}(-1)^{n-m-1}f^{(m)}(x)g^{(n-m-1)}(x)\Big]_{x=-1}^{x=1}$$

for all functions f and g of class $C^n([-1,1])$ with real values, and for all integers $n \in \mathbb{N}^*$.

2. *Define the following polynomials:*

$$R_n(x) = (x^2-1)^n \quad \text{and} \quad Q_n = R_n^{(n)}.$$

(a) *Show that for all $n \in \mathbb{N}$, Q_n is a polynomial of degree n and that $R_n^{(j)}(\pm 1) = 0$ for $0 \le j < n$.*

(b) *Deduce that the family $\{Q_n\}_{n\in\mathbb{N}}$ is orthogonal in H.*

(c) *Define $P_n = \frac{Q_n}{\|Q_n\|}$. Show that the sequence $\{P_n\}_{n\in\mathbb{N}}$ results from the sequence $\{x^n\}_{n\in\mathbb{N}}$ via the Gram-Schmidt orthonormalization process.*

(d) *Using Weierstrass's theorem, show that $\{P_n\}_{n\in\mathbb{N}}$ is a Hilbert basis of H.*

3. *Let $I_n = \int_{-1}^{1}(1-x^2)^n\,dx$.*

(a) *Show that for all $n \in \mathbb{N}^*$, we have $\|Q_n\|^2 = (2n)!I_n$.*

(b) *Show that for all $n \in \mathbb{N}^*$, we have $I_n = \frac{2n}{2n+1}I_{n-1}$.*

(c) *Deduce the value of I_n and that $\|Q_n\| = \frac{2^n n!}{\sqrt{n+\frac{1}{2}}}$.*

4. *Give the best approximation in $L^2([-1,1])$ of the function $f(x) = sgn(x)$ by a polynomial function of degree less than or equal to 3. Plot its graph.*

Exercice 3.43 (Laguerre Polynomials) *Let E be the Hilbert space defined by:*

$$E = \{f : [0,+\infty[\to \mathbb{R} : \int_0^{+\infty} |f(x)|^2\, e^{-x}\,dx < \infty\}.$$

For $n \in \mathbb{N}$, define:

$$p_n(x) = \frac{1}{n!}e^x\Big(\frac{d}{dx}\Big)^{(n)}(e^{-x}x^n).$$

1. *Compute the inner products $\langle x^m, p_n\rangle$, $n, m \in \mathbb{N}$.*

2. *Show that the functions p_n are polynomials and that they form an orthonormal system.*

3. *For $\alpha > 0$, compute $c_n = \langle e^{-\alpha x}, p_n \rangle$ and $\sum_{n=0}^{\infty} |c_n|^2$. Deduce that:*

$$e^{-\alpha x} = \sum_{n=0}^{\infty} c_n p_n(x)$$

is a convergent series in E.

4. *By performing the change of variable $t = e^{-kx}$, and applying Weierstrass's theorem, show that the family $\{e^{-kx}\}_{k\in\mathbb{N}}$ is a total system in E.*

5. *Deduce that $\{p_n\}_{n\in\mathbb{N}}$ is a Hilbert basis of E.*

Exercice 3.44 (Hermite Polynomials) *The Hermite polynomials H_n are defined by the relation:*

$$\left(\frac{d}{dx}\right)^{(n)} (e^{-x^2}) = (-1)^n H_n(x) e^{-x^2}$$

Recall that they satisfy the recurrence relation:

$$\begin{cases} H_0 = 1, \\ H_{n+1} = (-\frac{d}{dx} + 2x) H_n, \end{cases}$$

The Hermite functions ψ_n are defined by:

$$\psi_n = (\sqrt{\pi} 2^n n!)^{-\frac{1}{2}} H_n e^{-\frac{x^2}{2}}.$$

1. *Show that $\{\psi_n\}_{n\in\mathbb{N}}$ forms an orthonormal system in $L^2(\mathbb{R})$. To compute $\langle \psi_n, \psi_m \rangle$, one can integrate by parts p (or q) times.*

2. *Let g be a function in $L^2(\mathbb{R})$. Define, for $z \in \mathbb{C}$:*

$$G(z) = \int_{\mathbb{R}} e^{-itz} e^{-\frac{t^2}{2}} g(t)\, dt.$$

Show that the function G is holomorphic in the variable z and compute $\left(\frac{d}{dx}\right)^k G(0)$.

3. *Deduce that the family $\{\psi_n\}_{n\in\mathbb{N}}$ is a Hilbert basis of $L^2(\mathbb{R})$. Hint: use the injectivity of the Fourier transform.*