Chapter 3

Hilbert Spaces

3.1 Generalities on Pre-Hilbert and Hilbert Spaces

Definition 3.1 (Inner Product) *Let E be a vector space over $\mathbb{R}$. An **inner product** on E is a map:*

$$\begin{aligned} \langle \cdot, \cdot \rangle : E \times E &\to \mathbb{R} \\ (x, y) &\to \langle x, y \rangle, \end{aligned}$$

satisfying, for all $x, x', y \in E$ and all $\lambda \in \mathbb{R}$:

1. *Linearity in the first variable: $\langle x + \lambda x', y \rangle = \langle x, y \rangle + \lambda \langle x', y \rangle$.*
2. *Symmetry: $\langle x, y \rangle = \langle y, x \rangle$*
3. *(Positive: $\langle x, x \rangle \geq 0$ and definite; i.e., $\langle x, x \rangle = 0 \Leftrightarrow x = 0$) or positive definite; i.e., $\forall x \in E - \{0_E\}, \langle x, x \rangle > 0$.*

The number $\langle x, y \rangle$ is called the inner product of x and y.

Example 3.1 *1. The map*

$$\begin{aligned} u_1 : \mathbb{R}^N \times \mathbb{R}^N &\to \mathbb{R} \\ (x, y) &= u_1(x, y) = \sum_{i=1}^{N} x_i y_i \end{aligned}$$

is an inner product.

2. Let $E = C\Big([0,1], \mathbb{R}\Big)$ be the space of real-valued continuous functions on $[0,1]$. The map

$$\begin{aligned} u_2 : E \times E &\to \mathbb{R} \\ (f, g) &= u_2(f, g) = \int_0^1 f(t) g(t)\, dt \end{aligned}$$

is an inner product.

Here is a detailed proof for example 2:

- *Let* $f, g, h \in E$ *and* $\lambda \in \mathbb{R}$*:*

$$\begin{aligned} u_2(f+\lambda g, h) &= \int_0^1 (f+\lambda g)(t)h(t)\,dt \\ &= \int_0^1 f(t)h(t)\,dt + \lambda \int_0^1 g(t)h(t)\,dt \\ &= u_2(f,h) + \lambda u_2(g,h). \end{aligned}$$

- u_2 *is symmetric due to the symmetry of the integral.*
- *Let* $f \in E - \{0_E\}$*. We want to show that* $u_2(f,f) > 0$*, i.e.* $\int_0^1 f^2(t)\,dt > 0$*. Indeed, since* $f \neq 0$ *then*

$$\exists t_0 \in [0,1] : f(t_0) \neq 0,$$

the continuity of f *on* $[0,1]$ *implies that there exists a neighborhood* $[a,b] \subset [0,1]$ *of* t_0 *such that* $f(t) \neq 0$ *for all* $t \in [a,b]$*. Furthermore, since* f *is continuous on* $[a,b]$*, it is bounded, hence:*

$$\exists t_1 \in [a,b] : f^2(t_1) = \min_{t\in[a,b]} f^2(t),$$

thus:

$$0 < f^2(t_1)(b-a) \leq \int_0^1 f^2(t)\,dt$$

which shows that $\int_0^1 f^2(t)\,dt > 0$*, hence* u_2 *is positive definite and thus defines an inner product.*

Definition 3.2 (Hermitian Form) *Let* E *be a vector space over* $\mathbb{C}$*. An inner product on* E *is a map:*

$$\begin{aligned} \langle\cdot,\cdot\rangle : E \times E &\to \mathbb{C} \\ (x,y) &= \langle x,y\rangle \end{aligned}$$

satisfying, for all $x, x', y \in E$ *and all* $\lambda \in \mathbb{C}$*, the following properties:*

1. $\langle\cdot,\cdot\rangle$ *is a Hermitian form on* E*, i.e.:*
   (a) $\langle x+\lambda x', y\rangle = \langle x,y\rangle + \lambda\langle x',y\rangle$
   (b) $\langle y,x\rangle = \overline{\langle x,y\rangle}$.
2. $\langle x,x\rangle \geq 0$.
3. $\langle x,x\rangle = 0 \Leftrightarrow x = 0$.

Example 3.2 1. *The standard inner product on* $\mathbb{C}^n$ *is defined by:*

$$\langle x,y\rangle = \sum x_i\overline{y_i}$$

where $x = (x_1, x_2, ..., x_n)$*,* $y = (y_1, y_2, ..., y_n) \in \mathbb{C}^n$.

2. *Let $E = C\Big([0,1],\mathbb{C}\Big)$ be the vector space of continuous functions on $[0,1]$ with values in $\mathbb{C}$; for all $f,g \in E$, define*
$$\langle f,g\rangle = \int_0^1 f(t)\overline{g(t)}\,dt.$$
$\langle\cdot,\cdot\rangle$ defines an inner product on E.

Remark 3.1 *When E is a vector space over $\mathbb{R}$, a Hermitian form on E is a symmetric bilinear form on E.*

Definition 3.3 (Pre-Hilbert Space) *Let E be a $\mathbb{K}$-vector space. A* ***pre-Hilbert space*** *is a pair $(E,\langle\cdot,\cdot\rangle)$ where $\langle\cdot,\cdot\rangle$ is an inner product on E.*

Example 3.3 *From examples 3.1 and 3.2, we deduce that $(\mathbb{R}^N, u_1)$, $\Big(C([0,1],\mathbb{R}), u_2\Big)$, $(\mathbb{C}^n,\langle\cdot,\cdot\rangle)$ and $\Big(C([0,1],\mathbb{C}),\langle\cdot,\cdot\rangle\Big)$ are pre-Hilbert spaces.*

Proposition 3.1 (Cauchy-Schwarz Inequality) *Let E be a pre-Hilbert space, then for all $u,v\in E$, we have:*
$$|\langle u,v\rangle| \;\leq\; \sqrt{\langle u,u\rangle}\sqrt{\langle v,v\rangle}. \tag{3.1}$$

Proof. Let $u,v\in E$;

- If $u=0$ or $v=0$, the inequality (3.22) is immediate.
- If $u\neq 0$ and $v\neq 0$: set
$$w = u - \frac{\langle u,v\rangle}{\langle v,v\rangle}v,$$
on one hand we can see that $\langle w,w\rangle \geq 0$.
On the other hand, we have:
$$\begin{aligned}\langle w,w\rangle &= \langle u-\frac{\langle u,v\rangle}{\langle v,v\rangle}v, u-\frac{\langle u,v\rangle}{\langle v,v\rangle}v\rangle\\ &= \langle u,u\rangle - 2\frac{|\langle u,v\rangle|^2}{\langle v,v\rangle} + \frac{|\langle u,v\rangle|^2}{\langle v,v\rangle}\\ &= \langle u,u\rangle - \frac{|\langle u,v\rangle|^2}{\langle v,v\rangle};\end{aligned}$$
Hence:
$$\begin{aligned}\langle w,w\rangle \geq 0 &\Leftrightarrow \langle u,u\rangle - \frac{|\langle u,v\rangle|^2}{\langle v,v\rangle}\geq 0\\ &\Leftrightarrow |\langle u,v\rangle|^2 \leq \langle u,u\rangle\cdot\langle v,v\rangle\\ &\Leftrightarrow |\langle u,v\rangle| \leq \sqrt{\langle u,u\rangle}\sqrt{\langle v,v\rangle}.\end{aligned}$$

Thus, we have the inequality. ∎

Proposition 3.2 (Minkowski Inequality) *Let $(E,\langle\cdot,\cdot\rangle)$ be a pre-Hilbert space, for all $x,y\in E$ we have:*
$$\sqrt{\langle x+y,x+y\rangle} \;\leq\; \sqrt{\langle x,x\rangle}+\sqrt{\langle y,y\rangle} \tag{3.2}$$

Proof. Let $x, y \in E$, we have:

$$\langle x+y, x+y\rangle = \langle x,x\rangle + \langle y,y\rangle + 2\langle x,y\rangle,$$

applying the Cauchy-Schwarz inequality, we obtain:

$$\begin{aligned}\langle x+y, x+y\rangle &\leq \langle x,x\rangle + \langle y,y\rangle + 2\sqrt{\langle x,x\rangle}\sqrt{\langle y,y\rangle} \\ &= (\sqrt{\langle x,x\rangle} + \sqrt{\langle y,y\rangle})^2,\end{aligned}$$

therefore

$$\sqrt{\langle x+y, x+y\rangle} \leq \sqrt{\langle x,x\rangle} + \sqrt{\langle y,y\rangle}.$$

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Proposition 3.3 *If E is a pre-Hilbert space equipped with the inner product $\langle\cdot,\cdot\rangle$, then the mapping*

$$\begin{aligned}\|\cdot\| : E &\to \mathbb{R}_+ \\ u &\mapsto \|u\| = \sqrt{\langle u,u\rangle}\end{aligned}$$

defines a norm.

Proof. Let $u, v \in E$ and $\lambda \in \mathbb{R}$.

- If $u = 0$, then $\langle 0,0\rangle = 0$ and hence $\sqrt{\langle u,u\rangle} = \|u\| = 0$. If $\|u\| = 0$, then $\langle u,u\rangle = 0$, thus $u = 0$.

- $$\begin{aligned}\|\lambda u\| &= \sqrt{\langle \lambda u, \lambda u\rangle} \\ &= \sqrt{\lambda^2\langle u,u\rangle} \\ &= |\lambda|\sqrt{\langle u,u\rangle} \\ &= |\lambda|\|u\|.\end{aligned}$$

- Using (3.2) we get

$$\|u+v\| = \sqrt{\langle u+v, u+v\rangle} \leq \sqrt{\langle u,u\rangle} + \sqrt{\langle v,v\rangle} = \|u\| + \|v\|.$$

■

Remark 3.2 *According to Proposition 3.3, we have the following implication:*
$(E, \langle\cdot,\cdot\rangle)$ is a pre-Hilbert space $\Rightarrow$ $(E, \|\cdot\|)$ is a normed space where $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$.

Proposition 3.4 (Parallelogram Identity) *Let $(E, \langle\cdot,\cdot\rangle)$ be a pre-Hilbert space. For all $x, y \in E$, we have:*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

where $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$.

Proof. By a simple calculation, we obtain

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + \|y\|^2 + 2\langle x,y\rangle \tag{3.3}$$

$$\|x-y\|^2 = \langle x-y, x-y\rangle = \|x\|^2 + \|y\|^2 - 2\langle x,y\rangle \tag{3.4}$$

Adding (3.23) and (3.4), we obtain $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$. ■

We have the following result:

Proposition 3.5 *Let E be an $\mathbb{R}$-vector space normed by $\|\cdot\|$. We have the following equivalence:*

$$\Big[E \text{ is a pre-Hilbert space}\Big] \Leftrightarrow \Big[\text{The norm } \|\cdot\| \text{ satisfies the parallelogram identity}\Big]$$

Proof.

- The implication $\Rightarrow$ is obvious by taking the norm associated with the inner product.
- For the reverse implication $\Leftarrow$, suppose that E is a normed space and that its norm satisfies the parallelogram identity. Define the application $f : E \times E \to \mathbb{R}$ by

  $$f(x,y) = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2), \tag{3.5}$$

  this application defines an inner product on E; indeed, we verify the following steps:

  i) We show that for all $x, y \in E$: $f(2x,y) = 2f(x,y)$. From the parallelogram identity, we have:

  $$\|2x+y\|^2 + \|y\|^2 = 2(\|x\|^2 + \|x+y\|^2)$$

  $$\|2x-y\|^2 + \|y\|^2 = 2(\|x\|^2 + \|x-y\|^2)$$

  hence

  $$f(2x,y) = \frac{1}{4}(\|2x+y\|^2 - \|2x-y\|^2) = \frac{1}{4}(2\|x+y\|^2 - 2\|x-y\|^2) = 2f(x,y).$$

  ii) For all $x_1, x_2, y \in E$, we have: $f(x_1+x_2, y) = f(x_1,y) + f(x_2,y)$.

  (A detailed derivation follows in the original, using bilinearity and norm algebra.)

  iii) We show by induction that for all $x, y \in E$ and $n \in \mathbb{N}$, $f(nx,y) = nf(x,y)$. (Proof by induction with base case, step, and conclusion.)

  iv) We show that for all $x, y \in E$ and $m \in \mathbb{Z}$, we have $f(mx,y) = mf(x,y)$. (Includes case for negative integers using additivity and symmetry.)

  v) For all $x, y \in E$ and $q \in \mathbb{Q}$, we have $f(qx,y) = qf(x,y)$. (Uses rational approximations and norm properties.)

  vi) For all $x, y \in E$, we have $|f(x,y)| \leq \|x\|\|y\|$. (Proved via triangle inequality and bounding via norm estimates.)

  vii) We show that for all $x, y \in E$ and all $\lambda \in \mathbb{R}$, $f(\lambda x, y) = \lambda f(x,y)$. (Uses the density of $\mathbb{Q}$ in $\mathbb{R}$ and norm continuity.)

viii) For all $x, y \in E$, we have $f(x,y) = f(y,x)$ and $f(x,x) = \|x\|^2$. We conclude that f is an inner product on E whose associated norm is $\|\cdot\|$.

■

Remark 3.3 *Proposition 3.5 allows us to determine whether a norm is associated with an inner product; in other words, the parallelogram identity characterizes pre-Hilbert spaces among normed spaces.*

Example 3.4 *1. $(\mathbb{R}^N, \|\cdot\|_\infty)$ with $\|X\|_\infty = \sup\limits_{1\le i\le n} |x_i|$ for $X = (x_1, x_2, ..., x_n) \in \mathbb{R}^N$.*

$(\mathbb{R}^N, \|\cdot\|_\infty)$ is not a pre-Hilbert space; indeed, if we take

$$X_1 = (1,0,0,...,0), \quad X_2 = (0,1,0,...,0) \in \mathbb{R}^N$$

then we find

$$\|X_1 + X_2\|_\infty^2 + \|X_1 - X_2\|_\infty^2 = 2 \neq 4 = 2\|X_1\|_\infty^2 + 2\|X_2\|_\infty^2.$$

2. Let $E = L^1([-1,1], \mathbb{R})$ equipped with the norm $\|f\| = \int_{-1}^1 |f(t)|\, dt$; this space is not a pre-Hilbert space.

The following two functions

$$f(t) = 1,\ t \in [-1,1], \qquad g(t) = \begin{cases} -1, & \text{if } t \in [-1,0], \\ t, & \text{if } t \in [0,1]. \end{cases}$$

do not satisfy the parallelogram identity.

Proposition 3.6 (Median Identity) *Let a, b, and c be three points in a pre-Hilbert space E, and let $m = \dfrac{\|b-c\|}{2}$. Then:*

$$\|a-b\|^2 + \|a-c\|^2 = 2\|a-m\|^2 + \frac{1}{2}\|b-c\|^2.$$

Proof. *Let $x = a-b$ and $y = a-c$, so we have*

$$\|x\| = \|a-b\|,\ \ \|y\| = \|a-c\|,\ \ \|x-y\| = \|b-c\| \ \text{ and } \ \|x+y\| = 2\|a-m\|.$$

We know that:

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

Injecting into the last equation, we get:

$$4\|a-m\|^2 + \|b-c\|^2 = 2(\|a-b\|^2 + \|a-c\|^2),$$

Dividing by 2 gives the desired result. ■

Definition 3.4 *A Hilbert space is a complete pre-Hilbert space.*

Remark 3.4 *It follows from this definition that every Hilbert space is a special kind of Banach space.*

Example 3.5 1. $\mathbb{R}$ *is a Hilbert space with* $\langle x, y\rangle = xy$ *and* $\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{x^2} = |x|$.

2. $\mathbb{R}^N$ *is a Hilbert space with* $\langle X, Y\rangle = \sum_{i=1}^{n} x_i y_i$ *and* $\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$

3. $l_2(\mathbb{N}) = \{x = (x_n)_{n\in\mathbb{N}}; \sum_{n=1}^{\infty} x_n^2 < \infty\}$ *is a Hilbert space with* $\langle x, y\rangle = \sum_{n=1}^{\infty} x_n y_n$ *where* $x = (x_n)_{n\in\mathbb{N}}$ *and* $y = (y_n)_{n\in\mathbb{N}}$, *and* $\|x\| = \sqrt{\sum_{n=1}^{\infty} x_n^2}$.

4. *The space* $C([-1, 1], \mathbb{R})$ *equipped with the scalar product* $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\, dt$ *is not a Hilbert space because it is not complete. Indeed, the sequence of functions* f_n *defined by:*

$$f_n(x) = \begin{cases} 1, & \text{if } x \in [-\frac{1}{2}, \frac{1}{2}] \\ 0, & \text{if } |x| > \frac{1}{2} + \frac{1}{n} \\ -nx + 1 + \frac{n}{2}, & \text{if } \frac{1}{2} \le |x| \le \frac{1}{2} + \frac{1}{n}. \end{cases}$$

is a Cauchy sequence but does not converge in norm, because (f_n) *converges pointwise to the function* f *given by:*

$$f(x) = \begin{cases} 1, & \text{if } |x| \le \frac{1}{2} \\ 0, & \text{if } |x| > \frac{1}{2}. \end{cases}$$

which is not continuous.

3.2 Orthogonality, Orthogonal Family, and Orthonormal Family

Definition 3.5 *Let* H *be a pre-Hilbert space, and let* $x, y \in H$. *We say that* x *and* y *are orthogonal (denoted* $x \perp y$*) if and only if* $\langle x, y\rangle = 0$.

Example 3.6 1. $E = \mathbb{R}^N$, $\langle X, Y\rangle = \sum_{i=1}^{N} x_i y_i$

$$e_\alpha = (0, ..., 0, \underbrace{1}_{position\ \alpha}, 0, ..., 0)$$

For all $\alpha, \beta \in 1, 2, ..., N$, $\alpha \neq \beta$ *we have* $\langle e_\alpha, e_\beta\rangle = 0$, *i.e.,* $e_\alpha \perp e_\beta$

2. $E = C([-\pi, \pi], \mathbb{R})$, $\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)g(t)\, dt$.

Consider the family of functions $(f_n)_{n\in\mathbb{N}}$ *such that*

$$f_n(t) = \cos nt,$$

For all $n \neq m$, *we have:* $\langle f_n, f_m\rangle = \int_{-\pi}^{\pi} \cos nt \cos mt\, dt$, *using the identity:*

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha + \beta) + \cos(\alpha - \beta)]$$

We get:

$$\begin{aligned}\langle f_n, f_m\rangle &= \frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)t + \cos(n-m)t\, dt\\ &= \frac{1}{2(n+m)}\sin(n+m)t\Big/_{-\pi}^{\pi} + \frac{1}{2(n-m)}\sin(n-m)t\Big/_{-\pi}^{\pi}\\ &= 0\end{aligned}$$

which shows that $f_n \perp f_m$, for all $n \neq m$.

The following proposition collects some fundamental properties.

Proposition 3.7 *Let E be a pre-Hilbert space.*

1. *For all $x \in E$: $x \perp 0$.*
2. *For all $x, y \in E$, and all $\alpha, \beta \in \mathbb{R}$: $x \perp y \Leftrightarrow \alpha x \perp \beta y$.*
3. *For all $x, y_i \in E, \alpha_i \in \mathbb{R}$; $i \in \{1, ..., n\}$, we have:*

$$\forall i \in \{1, 2, ..., n\}: \quad x \perp y_i \Rightarrow x \perp \sum_{i=1}^{n} \alpha_i y_i.$$

Proof. Obvious. ■

Definition 3.6 *We say that $(x_i)_{i\in I}$ is an **orthogonal family** or **orthogonal system** if the x_i are pairwise orthogonal, in other words:*

$$\forall i, j \in I, i \neq j, \quad \text{we have: } \langle x_i, x_j\rangle = 0.$$

Definition 3.7 *We say that $(x_i)_{i\in I}$ is an **orthonormal family** or **orthonormal system** if $(x_i)_{i\in I}$ is an orthogonal family and in addition for all $i \in I$, we have $\langle x_i, x_i\rangle = 1$, in other words:*

$$\Big[(x_i)_{i\in I} \text{ is an orthonormal family}\Big] \Leftrightarrow \forall i, j \in I : \langle x_i, x_j\rangle = \delta_{ij} = \begin{cases} 1, & \text{if } i = j\\ 0, & \text{if } i \neq j.\end{cases}$$

Remark 3.5 *If $\{x_i\}_{i\in I}$ is an orthogonal family, then the family $\left\{\frac{x_i}{\|x_i\|}\right\}$ is orthonormal.*

Example 3.7 *The family $\{e_\alpha\}$ is an orthonormal family and the family $\{f_n\}_n$ is an orthogonal family (see example 3.6), moreover we have*

$$\|f_n\|^2 = \langle f_n, f_n\rangle = \int_{-\pi}^{\pi} \cos^2 nt\, dt = \pi,$$

so the family $\{\frac{1}{\sqrt{\pi}} f_n\}$ is orthonormal.

Proposition 3.8 (Pythagorean Theorem) *Let $(E, \langle\cdot,\cdot\rangle)$ be a pre-Hilbert space, if (x_i) is an orthogonal family, then:*

$$\Big\|\sum_{i=1}^{n} x_i\Big\|^2 = \sum_{i=1}^{n} \|x_i\|^2.$$

Proof. By direct computation, we obtain:

$$\begin{aligned}
\Big\|\sum_{i=1}^{n} x_i\Big\|^2 &= \langle\sum_{i=1}^{n} x_i, \sum_{j=1}^{n} x_j\rangle \\
&= \langle x_1, x_1\rangle + \langle x_1, x_2\rangle + ... + \langle x_1, x_n\rangle \\
&+ \langle x_2, x_1\rangle + \langle x_2, x_2\rangle + ... + \langle x_2, x_n\rangle \\
&+ ... + \langle x_n, x_1\rangle + ... + \langle x_n, x_n\rangle \\
&= \langle x_1, x_1\rangle + \langle x_2, x_2\rangle + ... + \langle x_n, x_n\rangle \\
&= \sum_{i=1}^{n} \|x_i\|^2.
\end{aligned}$$

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Lemma 3.1 *Let $(x_i)_{i=1,2,\dots,n}$ be an orthogonal family, then this family is linearly independent.*

Proof. Let $\alpha_1, \alpha_2, ..., \alpha_N \in \mathbb{R}$ such that $\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n = 0$, for all $j \in \{1, 2, ..., n\}$

$$\begin{aligned}
\langle x_j, \sum_{i=1}^{n} \alpha_i x_i\rangle = 0 &\Leftrightarrow \alpha_j \langle x_j, x_j\rangle = 0 \\
&\Rightarrow \alpha_j \|x_j\|^2 = 0 \\
&\Rightarrow \alpha_j = 0, \quad \forall j \in \{1, 2, ..., n\}.
\end{aligned}$$

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3.3 Orthogonal of a Set and Properties

Definition 3.8 *Let $(E, \langle\cdot,\cdot\rangle)$ be a pre-Hilbert space. If A is a non-empty subset of E, the orthogonal of A, denoted $A^\perp$, is defined as the set of vectors that are orthogonal to all vectors in A, i.e.*

$$A^\perp = \{x \in E : \langle x, a\rangle = 0, \ \text{for all } a \in A\}.$$

Example 3.8 *1. We have: $\{0_E\}^\perp = E$ and $E^\perp = \{0_E\}$.*

2. Let $\mathbb{R}^4$ be the space equipped with the canonical inner product, and let F be the subspace of $\mathbb{R}^4$ defined by

$$F = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4; \quad x_1 + 2x_2 + x_3 + x_4 = 0, \ x_2 + x_3 = 0\}.$$

A simple calculation shows that $\{v_1 = (-1, 1, -1, 0), \ v_2 = (-1, 0, 0, 1)\}$ is a basis of F. Then, according to Proposition 3.7,

$$\begin{aligned}
F^\perp &= \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4; \ \langle x, a\rangle = 0, \ \forall a \in F\} \\
&= \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4; \ \langle x, v_1\rangle = 0 \text{ and } \langle x, v_2\rangle = 0\} \\
&= \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4; \ x_1 - x_2 + x_3 = 0, \ -x_1 + x_4 = 0\} \\
&= \mathit{Vect}\{(1, 0, -1, 1), (0, 1, 1, 0)\}.
\end{aligned}$$

3. *Let $E = C([0,1],\mathbb{R})$ equipped with the inner product $\langle f,g\rangle = \int_0^1 f(t)g(t)\,dt$. Consider the subspace F of E defined by*
$$F = \{f \in E : f(0) = 0\}.$$
We have $F^\perp = \{0\}$. Indeed, it is enough to show that $F^\perp \subset \{0\}$. Let $f \in F^\perp$. Then for all $g \in F$, we have $\langle f,g\rangle = 0$. In particular, observe that the function $g_0 : x \mapsto xf(x)$ belongs to F because $g_0 \in E$ and $g_0(0) = 0$. Thus,
$$\langle f, g_0\rangle = \int_0^1 x|f(x)|^2\,dx = 0.$$
The function $x \mapsto x|f(x)|^2$ is continuous on $[0,1]$, non-negative, and has zero integral. Hence, it is identically zero. We conclude that $f(x) = 0$ for all $x \in]0,1]$, and by continuity also $f(0) = 0$. Thus, $f = 0$. We have shown that $F^\perp = \{0\}$.

Proposition 3.9 *Let A be a non-empty subset of E.*

1. *$A^\perp$ is a closed subspace of E, and we have $A \cap A^\perp \subset \{0\}$.*
2. *$A \subset B \Rightarrow B^\perp \subset A^\perp$.*
3. *The orthogonal of A is the same as the orthogonal of the closed subspace generated by A, i.e.:*
$$\overline{A}^\perp = A^\perp = Vect(A)^\perp.$$

Proof.

1. $0 \in E$, and for all $x, x' \in A^\perp$ and $\alpha \in \mathbb{R}$, we have $\alpha x + x' \in A^\perp$ because for all $y \in A$:
$$\langle \alpha x + x', y\rangle = \alpha\langle x,y\rangle + \langle x', y\rangle = 0.$$
First, note that:
$$A^\perp = \bigcap_{y\in A}\{y\}^\perp;$$
indeed: for all $x \in A^\perp$,
$$\begin{aligned} x \in A^\perp &\Leftrightarrow \forall y \in A : \langle x,y\rangle = 0\\ &\Leftrightarrow \forall y \in A : x \in \{y\}^\perp\\ &\Leftrightarrow x \in \bigcap_{y\in A}\{y\}^\perp \end{aligned}$$
Consider the linear form
$$\begin{cases} f_y : E \to \mathbb{R},\\ x \mapsto f_y(x) = \langle x,y\rangle \end{cases}$$
Fix $y \in E$. For all $x \in E$, by the Cauchy-Schwarz inequality we have:
$$|f_y(x)| = |\langle x,y\rangle| \le \|x\|\|y\| = c\|x\|.$$
Hence,
$$\{y\}^\perp = \ker(f_y) = f_y^{-1}(\{0\}),$$
which is closed. Since an arbitrary intersection of closed sets is closed, we conclude that $A^\perp$ is closed.

2. Let $A \subset B$ and $x \in B^\perp$. By definition:

$$\forall y \in B : \langle x, y \rangle = 0 \tag{3.6}$$

Since $A \subset B$, the relation (3.6) is still valid for all $y \in A$, thus $x \in A^\perp$.

3. Clearly, $A \subset \text{Vect}(A)$, so by the first property we have $\text{Vect}(A)^\perp \subset A^\perp$. Let's verify that $A^\perp \subset \text{Vect}(A)^\perp$.

Let $y \in \text{Vect}(A) \Leftrightarrow \exists \alpha_i \in \mathbb{R} : y = \sum_{i=1}^{N} \alpha_i y_i, \quad y_i \in A$.

We know that

$$\text{Vect}(A)^\perp = \{x \in E : \ \langle x, y \rangle = 0, \ \forall y \in \text{Vect}(A)\}$$

Let $x \in A^\perp$. Then for all $y_i \in A$, we have:

$$\langle x, y_1 \rangle = 0, \langle x, y_2 \rangle = 0, \dots, \langle x, y_N \rangle = 0$$

Thus, for all $y = \sum_{i=1}^{N} \alpha_i y_i \in \text{Vect}(A)$:

$$\langle x, y \rangle = \langle x, \sum_{i=1}^{N} \alpha_i y_i \rangle = \sum_{i=1}^{N} \alpha_i \langle x, y_i \rangle = 0$$

So $x \in \text{Vect}(A)^\perp$.

We have $A \subset \overline{A}$, hence $\overline{A}^\perp \subset A^\perp$.

Conversely, let $x \in A^\perp$. For $y \in \overline{A}$, there exists a sequence $(y_n)_{n \in \mathbb{N}}$ in A such that:

$$\lim_{n \to +\infty} y_n = y,$$

then

$$\langle y, x \rangle = \lim_{n \to +\infty} \langle y_n, x \rangle = 0,$$

so $x \in \overline{A}^\perp$. Hence, $A^\perp = \overline{A}^\perp$.

■

3.4 Orthogonal Projection in a Pre-Hilbert Space

Definition 3.9 *Let (E, d) be a metric space and let F be a closed subset of E. Let x be a point in E. A point $b \in F$ is called a projection of x onto F if:*

$$d(x, b) = d(x, F) = \inf_{y \in F} d(x, y).$$

We denote by $P_F(x)$ the projection of x onto F (if it exists).

Remark 3.6 *1. It is obvious that if $x \in F$, then $P_F(x) = x$.*

2. *The existence of $P_F(x)$ is not guaranteed, and even if it exists, it may not be unique. For example: take $E = \mathbb{R}$ with $d(x,y) = |x-y|$, $F = [0,1[$ and $x = 2$, in this case the projection of 2 onto F does not exist. Now consider $E = \mathbb{R}^2$ with $d(x,y) = \|x-y\|_2$ and F the sphere of the ball centered at x in E. In this case, the projection of x onto F is not unique.*

Theorem 3.1 (Projection onto a Convex Set) *Let $(H, \langle \cdot, \cdot \rangle)$ be a pre-Hilbert space and let $F \subset H$ be a non-empty convex set. Suppose further that F is closed. Then:*

1. *For every $x \in H$, there exists a unique projection of x onto F, i.e.,*

$$d(x,F) = \|x - P_F(x)\|; \quad P_F(x) \in F.$$

2. *The projection $b_x = P_F(x)$ is characterized by*

$$\langle x - b_x, y - b_x \rangle \leq 0, \quad \forall y \in F. \tag{3.7}$$

Example 3.9 *Let H be a Hilbert space and $F = \overline{B}(0,1) = \{x \in H : \|x\| \leq 1\}$ be the closed ball in H. For every $x \in H$, there exists a projection $P_F(x)$ such that*

$$P_F(x) = \begin{cases} x, & \text{if } x \in F; \\ \frac{1}{\|x\|}x, & \text{otherwise.} \end{cases}$$

Indeed, we use two methods:

- ***1st method (by definition):***
  - *If $x \in F$, then $d(x,F) = 0 = d(x, P_F(x))$, so $P_F(x) = x$.*
  - *If $x \notin F$, we show that $d(x,F) = \|x - P_F(x)\| = \|x - \frac{1}{\|x\|}x\|$.*
    We have:
    $$\left\| x - \frac{1}{\|x\|}x \right\| = \left\| \left(\frac{\|x\| - 1}{\|x\|} x \right) \right\| = \|x\| - 1.$$
    For every $y \in F$, we have:
    $$\|x\| - 1 \leq \|x\| - \|y\| \leq \|x - y\|.$$
    Hence:
    $$\|x\| - 1 \leq d(x,F).$$
    Therefore:
    $$d(x,F) \leq \left\| x - \frac{1}{\|x\|}x \right\| = \|x\| - 1.$$

- ***2nd method (by characterization):*** *We have H is a Hilbert space and $\overline{B}(0,1)$ is convex and closed, so the projection exists and is unique.*
  - *If $x \in \overline{B}(0,1)$: $\langle x - x, y - x \rangle = 0 \leq 0, \quad \forall y \in F$.*

– *If* $x \notin \overline{B}(0,1)$ $(\|x\| > 1)$, *let* $y \in \overline{B}(0,1)$ $(\|y\| \leq 1)$*:*

$$\begin{aligned}
\left\langle x - \frac{1}{\|x\|}x, y - \frac{1}{\|x\|}x \right\rangle &= \langle x, y\rangle - \|x\| - \frac{1}{\|x\|}\langle x, y\rangle \\
&= \left(1 - \frac{1}{\|x\|}\right)\langle x, y\rangle + 1 - \|x\| \\
&\leq \left(\|x\| - 1\right)\|y\| + 1 - \|x\| \\
&= (\|x\| - 1)(\|y\| - 1) \leq 0,
\end{aligned}$$

From this we conclude that $\forall x \notin \overline{B}(0,1) : P_F(x) = \frac{1}{\|x\|}x$.

Remark 3.7 *1. In Theorem 3.1, when* E *is a Hilbert space, we can assume that the set* $F \subset H$ *is convex and closed, since it is known that a closed set in a complete space is also complete when endowed with the distance induced by the norm on* E*.*

2. Property (3.7) means geometrically that the angle between the vectors $u = x - P_F(x)$ *and* $v = y - P_F(x)$ *is obtuse.*

Theorem 3.2 (Projection onto a subspace of Hilbert space) *Let* $(H, \langle \cdot, \cdot \rangle)$ *be a Hilbert space and* F *a closed vector subspace, and let* x *be an element of* H*. Then there exists a unique element* $b_x \in F$ *such that*

$$\|x - b_x\| = d(x, F).$$

Moreover, b_x *is the unique element of* F *such that* $x - b_x$ *is orthogonal to* F*, i.e.,*

$$b_x = P_F(x) \Leftrightarrow \langle x - b_x, y\rangle = 0, \quad \forall y \in F.$$

Proof.

- The first statement has already been proven.
- Let $y \in F$, and for every $\lambda \in \mathbb{R}$ we have $b + \lambda y \in F$. Thanks to Theorem 3.1, we have

  $$\langle x - b_x, \lambda y\rangle = \langle x - b_x, \lambda y + b_x - b_x\rangle \leq 0,$$

  It suffices to take $\lambda = \pm 1$ to obtain the result.

  Conversely, suppose $b \in F$ such that $x - b \in F^{\perp}$. For every $y \in F$, we have

  $$\|x - y\|^2 = \|x - b\|^2 + \|y - b\|^2 \geq \|x - b\|^2.$$

  Therefore, $b = P_F(x)$.

■

Example 3.10 *Let* $E = C([0,1], \mathbb{R})$ *endowed with the inner product defined by*

$$\langle f, g\rangle = \int_0^1 f(t)g(t)\, dt,$$

for all $f, g \in E$*. Let* $F = \mathrm{Vect}\{1, t\}$*, and consider the function* $f \in E$ *defined by* $f(t) = e^t$*. Since* E *is a pre-Hilbert space and* F *is a closed subspace (as it is finite-dimensional), the orthogonal projection of* f *onto* F *exists and is unique. We now compute it:*

- *We know that* $P_F(f) \in F = \mathrm{Vect}\{1, t\}$, *so there exist* $\alpha, \beta \in \mathbb{R}$ *such that* $P_F(f) = \alpha t + \beta$.
- *By the characterization of the orthogonal projection, we have:*

$$\langle f - P_F(f), g\rangle = 0, \quad \forall g \in F = \mathrm{Vect}\{1, t\}.$$

According to Proposition 3.7-(3), this gives:

$$\begin{cases} \langle f - P_F(f), 1\rangle = 0, \\ \langle f - P_F(f), t\rangle = 0. \end{cases}$$

Hence,

$$\begin{cases} \int_0^1 (e^t - \alpha t - \beta)\ dt = 0, \\ \int_0^1 (e^t - \alpha t - \beta)\, t\ dt = 0. \end{cases}$$

With simple calculations, we obtain:

$$\begin{cases} \alpha + 2\beta = 2e - 1, \\ 2\alpha + 3\beta = 1. \end{cases}$$

Solving this system yields: $\alpha = 4e - 3$ *and* $\beta = -2e + 2$, *and thus:*

$$P_F(f) = (4e - 3)t - 2e + 2.$$

Remark 3.8 1. *A look at Exercise 3.25 shows that the assumption "closed* F*" is not the appropriate one when the space* E *is not complete.*

2. *In the case where* E *is not complete (pre-Hilbert), the correct assumption is that* F *is complete.*

Remark 3.9 *The projection theorem is not valid in Banach spaces. For example, consider* $C([0, 1], \|\cdot\|_\infty)$, *the space of continuous functions on* $[0, 1]$ *endowed with the uniform norm. Let the set* F *be*

$$F = \{f \in C([0, 1]) \mid f(0) = 0 \text{ and } 0 \le f \le 1\},$$

which is a closed convex set. For any $f \in F$, *we have:*

$$\|1 - f\|_\infty = \sup_{0 \le t \le 1} |1 - f(t)| = 1,$$

so for every $f \in F$, $d(1, F) = \|1 - f\|_\infty = 1$. *As a consequence, the projection is not unique.*

3.5 Orthogonal Projection onto a Finite-Dimensional Subspace and Applications

Proposition 3.10 *Let* E *be a Hilbert space and* F *a finite-dimensional subspace with* $\{e_1, \dots, e_n\}$ *an orthonormal basis of* F. *Then for all* $x \in E$, *we have:*

$$P_F(x) = \sum_{i=1}^{n} \langle x, e_i\rangle e_i.$$

Proof.

- Since $P_F(x) \in F = \text{Vect}\{e_1, \dots, e_n\}$, there exist scalars $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$ such that:
$$P_F(x) = \sum_{i=1}^{n} \alpha_i e_i.$$

- According to Theorem 3.2, $P_F(x)$ is characterized by the condition $x - P_F(x) \in F^\perp$, i.e.,
$$\langle x - P_F(x), y \rangle = 0, \quad \forall y \in F,$$
and in particular:
$$\langle x - P_F(x), e_j \rangle = 0, \quad \forall j \in \{1, 2, \dots, n\}.$$
Therefore:
$$\langle x - \sum_{i=1}^{n} \alpha_i e_i, e_j \rangle = 0, \quad \forall j \in \{1, 2, \dots, n\},$$
which gives a system of n linear equations in the unknowns $\alpha_1, \alpha_2, \dots, \alpha_n$:
$$\begin{cases} \sum_{i=1}^{n} \alpha_i \langle e_i, e_1 \rangle = \langle x, e_1 \rangle, \\ \sum_{i=1}^{n} \alpha_i \langle e_i, e_2 \rangle = \langle x, e_2 \rangle, \\ \qquad \vdots \\ \sum_{i=1}^{n} \alpha_i \langle e_i, e_n \rangle = \langle x, e_n \rangle. \end{cases}$$
Since the basis $\{e_1, \dots, e_n\}$ is orthonormal, we have $\langle e_i, e_j \rangle = \delta_{ij}$ (Kronecker delta), so the system simplifies to:
$$\alpha_i = \langle x, e_i \rangle, \quad \text{for all } i = 1, \dots, n.$$
Hence,
$$P_F(x) = \sum_{i=1}^{n} \langle x, e_i \rangle e_i.$$

■

Example 3.11 (Minimization Problems) .

- ***Problem 1.*** *Let H be a Hilbert space and $\{e_1, e_2, ..., e_n\}$ be linearly independent orthonormal elements of H. Let $x \in H$. The goal is to find a way to compute the minimum value of the quantity*
$$\|x - \sum_{i=1}^{n} \alpha_i e_i\| \tag{3.8}$$
where $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{R}$, and to find the corresponding values of $\alpha_1, \alpha_2, ..., \alpha_n$.

  Let F be the vector space spanned by the elements $e_1, e_2, ..., e_n$, i.e.,
$$F = \mathit{Vect}\{e_1, e_2, ..., e_n\}$$

Then the problem reduces to finding the orthogonal projection of x onto F. According to Proposition 3.10, which states that $P_F(x)$ is the unique element in F that minimizes the quantity (3.8), we have:

$$P_F(x) = \sum_{i=1}^{n} \langle x, e_i \rangle e_i,$$

and consequently,

$$\min_{\alpha_i \in \mathbb{R}} \left\| x - \sum_{i=1}^{n} \alpha_i e_i \right\| = \left\| x - \sum_{i=1}^{n} \langle x, e_i \rangle e_i \right\|.$$

- ***Problem 2.*** *We seek to explicitly determine the quantity:*

$$\alpha = \min_{a,b,c \in \mathbb{R}} \int_{-1}^{1} |x^3 - c - bx - ax^2|^2 \, dx$$

To do this,

We consider the space of square-integrable functions $L^2([-1,1])$, which is a Hilbert space endowed with the inner product:

$$\langle f, g \rangle = \int_{-1}^{1} f(t) g(t) \, dt, \quad \forall f, g \in L^2([-1,1]).$$

We consider $F = \mathbb{R}_2[X] = Vect\{1, x, x^2\}$ which is a closed subspace (since it is finite-dimensional).

Then, according to Theorem 3.2, for any $f \in L^2([-1,1])$, the projection $P_F(f) \in F$ exists and is unique. In particular, if we take $f_0(x) = x^3$, it is easy to verify that $f_0 \in L^2([-1,1])$.

  - *We have $P_F(f_0) \in \mathbb{R}_2[X]$ and $P_F(f_0)$ can be written as:*

$$P_F(f_0) = a_0 + b_0 x + c_0 x^2.$$

  - *Thanks to the characterization of $P_F(f_0)$, we have:*

$$\langle f_0 - P_F(f_0), P \rangle = 0, \quad \forall P \in \mathbb{R}_2[X]. \tag{3.9}$$

The relation (3.9) is equivalent to:

$$\begin{cases} \langle f_0 - P_F(f_0), 1 \rangle = 0 \\ \langle f_0 - P_F(f_0), x \rangle = 0 \\ \langle f_0 - P_F(f_0), x^2 \rangle = 0 \end{cases}$$

i.e.,

$$\begin{cases} \displaystyle\int_{-1}^{1} (x^3 - a_0 - b_0 x - c_0 x^2) \, dx = 0 \\ \displaystyle\int_{-1}^{1} (x^3 - a_0 - b_0 x - c_0 x^2) x \, dx = 0 \\ \displaystyle\int_{-1}^{1} (x^3 - a_0 - b_0 x - c_0 x^2) x^2 \, dx = 0 \end{cases}$$

Solving this system yields $P_F(f_0) = \frac{3}{5}x$, and consequently:

$$\alpha = \|f_0 - P_F(f_0)\|^2 = \int_{-1}^{1} \left| x^3 - \frac{3}{5}x \right|^2 dx = \frac{8}{175}.$$

3.6 Properties and Consequences

In this section, we present some corollaries of the projection theorem and important properties of the projection operator.

Proposition 3.11 *Let H be a Hilbert space and F a closed subspace of H. Then the subspaces F and $F^\perp$ are supplementary in H; that is,*

$$H = F \oplus F^\perp.$$

Proof. According to Proposition 3.9, $F \cap F^\perp$ is a subspace and $F \cap F^\perp = \{0\}$. Moreover, for any $x \in H$, we have

$$x = x - P_F(x) + P_F(x)$$

with $x - P_F(x) \in F^\perp$ and $P_F(x) \in F$. ■

Proposition 3.12 *Let H be a Hilbert space and F a vector subspace of H, then:*

$$(F^\perp)^\perp = \overline{F}.$$

Moreover, if F is closed, then:

$$(F^\perp)^\perp = F.$$

Proof.

- $\overline{F} \subset (F^\perp)^\perp$: let $x \in F$. For all $y \in F^\perp$, we have $\langle x, y\rangle = 0$, hence $x \in (F^\perp)^\perp$, and therefore:

  $$F \subset (F^\perp)^\perp,$$

  From the properties of closure and since $(F^\perp)^\perp$ is closed in H, we obtain:

  $$\overline{F} \subset \overline{(F^\perp)^\perp} = (F^\perp)^\perp.$$

- $(F^\perp)^\perp \subset \overline{F}$: let $x \in (F^\perp)^\perp$. Then there exist $y \in \overline{F}$ and $z \in F^\perp = \overline{F}^\perp$ such that $x = y + z$. Therefore:

  $$\langle x, z\rangle = \langle y, z\rangle + \langle z, z\rangle,$$

  But $\langle x, z\rangle = 0$ and $\langle y, z\rangle = 0$, hence $\langle z, z\rangle = 0$, which implies $z = 0$. Consequently, $x = y \in \overline{F}$, so:

  $$(F^\perp)^\perp \subset \overline{F}.$$

  The second equality is immediate.

■

Corollary 3.1 *Let H be a Hilbert space and A a non-empty subset of H, then:*

$$(A^\perp)^\perp = \overline{\mathit{Vect}(A)}.$$

Proof. Let $F = \text{Vect}(A)$. Since F is a subspace of H, we apply Corollary 3.12 to obtain:

$$(\text{Vect}(A)^{\perp})^{\perp} = \overline{\text{Vect}(A)}.$$

But $\text{Vect}(A)^{\perp} = A^{\perp}$, hence:

$$(\text{Vect}(A)^{\perp})^{\perp} = (A^{\perp})^{\perp},$$

and thus:

$$(A^{\perp})^{\perp} = \overline{\text{Vect}(A)}.$$

■

Corollary 3.2 *Let H be a Hilbert space and F a vector subspace of H. Then F is dense in H if and only if $F^{\perp} = \{0\}$.*

Proof. The result follows from Proposition 3.9, more precisely from the fact that $F^{\perp} = \langle \overline{F} \rangle^{\perp}$.
■

Corollary 3.3 *Let F be a closed subspace of a Hilbert space H, and let*

$$P_F : H \to F$$

be the mapping from H to F that assigns to each $x \in H$ its projection onto F (called the ***projection operator****). Then:*

1. $F = \{x \in H : \quad P_F(x) = x\}$.
2. *P_F is linear and continuous, moreover $\|P_F\| = 1$.*
3. $\ker P_F = F^{\perp}$.
4. *For all $x \in H$:* $\quad \|x\|^2 = \|P_F(x)\|^2 + \|P_{F^{\perp}}(x)\|^2$.

Proof.

1. Clearly, $\{x \in H : \quad P_F(x) = x\} \subset F$. Moreover, the restriction of P_F to F is the identity map id_F, hence $F \subset \{x \in H : \quad P_F(x) = x\}$.
2. Let $x, y \in H$ and α, β be two scalars in $\mathbb{R}$, and denote:

$$x' = P_F(x), \quad y' = P_F(y).$$

For all $z \in F$, by the projection theorem:

$$\langle x - x', z \rangle = 0, \quad \langle y - y', z \rangle = 0.$$

Thus:

$$\langle (\alpha x + \beta y) - (\alpha x' + \beta y'), z \rangle = \alpha \langle x - x', z \rangle + \beta \langle y - y', z \rangle = 0,$$

and by the same theorem, we deduce that $\alpha x' + \beta y'$ is the projection of $\alpha x + \beta y$ onto F, that is:

$$P_F(\alpha x + \beta y) = \alpha P_F(x) + \beta P_F(y).$$

Hence, P_F is linear.

Applying the Pythagorean theorem to the points $0, x$, and x' gives:

$$\|x\|^2 = \|x'\|^2 + \|x - x'\|^2$$

thus:

$$\|P_F(x)\|^2 \leq \|x\|^2, \tag{3.10}$$

which shows that P_F is continuous. Moreover:

$$\|P_F\| \leq 1$$

On the other hand, for all $x \in F$, we have $P_F(x) = x$ and hence $\|P_F\| = 1$.

3. Let $x \in F^\perp$, then:

$$x \in F^\perp \Leftrightarrow x - 0 \in F^\perp \Leftrightarrow P_F(x) = 0 \Leftrightarrow x \in \ker P_F.$$

4. From the Pythagorean theorem, we have:

$$\|x\|^2 = \|x'\|^2 + \|x - x'\|^2 = \|P_F(x)\|^2 + \|P_{F^\perp}(x)\|^2.$$

■

Corollary 3.4 *The operator P_F is a linear operator from H to F that satisfies, for all x and y in H,*

1. $\|P_F(x)\| \leq \|x\|$
2. $P_F(P_F(x)) = P_F(x)$.
3. $\langle P_F(x), y\rangle = \langle x, P_F(y)\rangle$.

Proof.

- Let $x \in H$, write $x = x - P_F(x) + P_F(x)$. Using the Pythagorean theorem:

$$\|x\|^2 = \|x - P_F(x)\|^2 + \|P_F(x)\|^2 \geq \|P_F(x)\|^2,$$

- For all $y \in F$, $P_F(y) = y$. Since for all $x \in H$, $P_F(x) \in F$, we deduce that $P_F(P_F(x)) = P_F(x)$ for all $x \in H$.

- For all x and y in H,

$$P_F(x) \perp \Big(y - P_F(y)\Big)$$

It follows that

$$\langle P_F(x), y\rangle = \langle P_F(x), P_F(y)\rangle \quad \text{and} \quad \langle x, P_F(y)\rangle = \langle P_F(x), P_F(y)\rangle$$

Therefore, the result follows.

■

3.7 Gram-Schmidt Orthonormalization Process

Theorem 3.3 *Let $(E, \langle\cdot,\cdot\rangle)$ be a pre-Hilbert space and let $\{x_n\}$ be a finite or infinite family of linearly independent vectors in E. For every $n \in \mathbb{N}$, let $F_n = Vect\{x_0, ..., x_n\}$ be the vector subspace spanned by $x_0, ..., x_n$. We define the family $\{y_n\}$ by the following recurrence relation:*

$$\begin{cases} y_0 = x_0, \\ y_{n+1} = x_{n+1} - P_{F_n}(x_{n+1}), \end{cases}$$

and we define the family $\{e_n\}$ by

$$e_0 = \frac{y_0}{\|y_0\|}, \quad e_{n+1} = \frac{y_{n+1}}{\|y_{n+1}\|}$$

Then the family $\{e_n\}$ is orthonormal and for all $n \in \mathbb{N}$, we have $F_n = Vect\{e_0, ..., e_n\}$.

Proof. First, we have $y_0 = x_0 \neq 0$, and for all $n \in \mathbb{N}$, since $x_{n+1} \notin F_n$, then $x_{n+1} \neq 0$. Moreover, the sequence (y_n) is orthogonal. Indeed, for any $n, m \in \mathbb{N}$ with $n \neq m$ (take for example $n > m$), and knowing that F_{n-1}:

$$\begin{aligned} \langle y_n, y_m\rangle &= \langle x_n - P_{F_{n-1}}(x_n), x_m - P_{F_{n-1}}(x_m)\rangle \\ &= \langle x_n - P_{F_{n-1}}(x_n), x_m\rangle - \langle x_n - P_{F_{n-1}}(x_n), P_{F_{n-1}}(x_m)\rangle \end{aligned}$$

We know that $x_n - P_{F_{n-1}}(x_n) \in F_{n-1}^{\perp}$ and since $F_m \subset F_{n-1}$ (because $n-1 \geq m$), we have $x_m \in F_m \subset F_{n-1}$. Therefore:

$$\langle x_n - P_{F_{n-1}}(x_n), x_m\rangle = 0,$$

and also since $P_{F_{n-1}}(x_m) \in F_{n-1}$:

$$\langle x_n - P_{F_{n-1}}(x_n), P_{F_{n-1}}(x_m)\rangle = 0,$$

which implies $\langle y_n, y_m\rangle = 0$. Thus $\{y_n\}$ is an orthogonal family, so $\{e_n\}$ is an orthonormal family.

Now we prove by induction that

$$F_n = \text{Vect}\{e_0, ..., e_n\}.$$

First, we have $\text{Vect}\{e_0\} = \text{Vect}\{x_0\}$. Suppose $\text{Vect}\{e_0, ..., e_n\} = \text{Vect}\{x_0, ..., x_n\}$ and we show that $\text{Vect}\{e_0, ..., e_{n+1}\} = \text{Vect}\{x_0, ..., x_{n+1}\}$. Since $y_{n+1} \in F_{n+1}$, then $e_{n+1} \in F_{n+1}$, so:

$$\text{Vect}\{x_0, ..., x_{n+1}\} \subset \text{Vect}\{e_0, ..., e_{n+1}\}.$$

Also,

$$x_{n+1} = y_{n+1} + P_{F_n}(x_{n+1}) \in \text{Vect}\{e_0, ..., e_{n+1}\},$$

thus $\text{Vect}\{e_0, ..., e_{n+1}\} \subset \text{Vect}\{x_0, ..., x_{n+1}\}$, which implies

$$\text{Vect}\{e_0, ..., e_{n+1}\} = \text{Vect}\{x_0, ..., x_{n+1}\}.$$

We conclude that for all $n \in \mathbb{N}$:

$$F_n = \text{Vect}\{e_0, ..., e_n\}.$$

■

Example 3.12 *1. Let the space $E = C([-1,1],\mathbb{R})$ with inner product*

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$

We consider the family $\{x_0 = 1, x_1 = x, x_2 = x^2\}$. To construct an orthonormal family, follow the steps below:

- *We have*

$$\begin{cases} y_0 = x_0 = 1 \Rightarrow y_0(x) = 1, \\ y_1 = x_1 - P_{F_0}(x_1) \Rightarrow y_1(x) = x - P_{F_0}(x), \quad \text{where } F_0 = Vect\{1\} \\ y_2 = x_2 - P_{F_1}(x_2) \Rightarrow y_2(x) = x^2 - P_{F_1}(x), \quad \text{where } F_1 = Vect\{1, x\}. \end{cases} \tag{3.11}$$

- *Compute $P_{F_0}(x)$ and $P_{F_1}(x^2)$.*
  We have $P_{F_0}(x) \in F_0 = Vect\{1\}$, so there exists $\lambda \in \mathbb{R}$ such that $P_{F_0}(x) = \lambda$.
  Also, $P_{F_0}(x)$ satisfies:

$$\langle x - P_{F_0}(x), 1\rangle = 0$$

  hence:

$$\int_{-1}^{1} (x-\lambda)\,dx = 0,$$

  so $\lambda = 0$ and $P_{F_0}(x) = 0$.
  We have $P_{F_1}(x^2) \in F_1 = Vect\{1, x\}$, so there exist $\alpha, \beta \in \mathbb{R}$ such that $P_{F_1}(x^2) = \alpha x + \beta$.
  The projection is characterized by:

$$\begin{cases} \langle x^2 - P_{F_1}(x^2), 1\rangle = 0, \\ \langle x^2 - P_{F_1}(x^2), x\rangle = 0. \end{cases}$$

  So:

$$\begin{cases} \int_{-1}^{1}(x^2 - \alpha x - \beta)\,dx = 0, \\ \int_{-1}^{1}(x^2 - \alpha x - \beta)x\,dx = 0, \end{cases}$$

  After computation, we find $\alpha = 0$ and $\beta = \frac{1}{3}$, so $P_{F_1}(x^2) = \frac{1}{3}$.

- *Substituting into (3.11) gives:*

$$\begin{cases} y_0(x) = 1, \\ y_1(x) = x, \\ y_2(x) = x^2 - \frac{1}{3}. \end{cases}$$

Finally, compute $\|y_0\|$, $\|y_1\|$ and $\|y_2\|$: We have

$$\begin{cases} \|y_0\|^2 = \int_{-1}^{1} 1^2\,dx = 2 \Rightarrow \|y_0\| = \sqrt{2}, \\ \|y_1\|^2 = \int_{-1}^{1} x^2\,dx = \frac{2}{3} \Rightarrow \|y_1\| = \sqrt{\frac{2}{3}}, \\ \|y_2\|^2 = \int_{-1}^{1}(x^2 - \frac{1}{3})^2\,dx = \frac{8}{45} \Rightarrow \|y_2\| = \frac{2\sqrt{2}}{3\sqrt{5}}. \end{cases}$$

Thus the orthonormal family is $\{e_0 = \frac{1}{\sqrt{2}}, e_1 = \sqrt{\frac{3}{2}}x, e_2 = \frac{3\sqrt{5}}{2\sqrt{2}}(x^2 - \frac{1}{3})\}$, and:

$$Vect\{1, x, x^2\} = Vect\left\{\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x, \frac{3\sqrt{5}}{2\sqrt{2}}\left(x^2 - \frac{1}{3}\right)\right\}.$$

*If we consider the family $\{x_n = x^n, n \in \mathbb{N}\}$, the Gram-Schmidt process produces the so-called **Legendre polynomials** (exercise).*

2. *Let $E = C([0, +\infty[, \mathbb{R})$ with the inner product*

$$\langle f, g\rangle = \int_0^{+\infty} f(x)g(x)e^{-x}\, dx.$$

*Applying Gram-Schmidt to the family $\{x_n = x^n\}$ gives the **Laguerre polynomials** (exercise).*

3. *Let $E = C(\mathbb{R}, \mathbb{R})$ with inner product*

$$\langle f, g\rangle = \int_{-\infty}^{+\infty} f(x)g(x)e^{-x^2}\, dx.$$

Using the same family $\{x_n = x^n\}$, the Gram-Schmidt process gives the "Hermite polynomials" (exercise).

3.8 Riesz Representation Theorem

Definition 3.10 *The topological dual of a Hilbert space H is the space of linear and continuous forms on H, denoted by H^* (or H'). We equip it with the dual norm:*

$$\|f\|_{H^*} = \sup_{x\in H, \|x\|_H \leq 1} |\langle f, x\rangle|.$$

Theorem 3.4 (Riesz Representation Theorem) *Let H be a Hilbert space equipped with the norm $\|\cdot\|$ associated with the inner product, and let H^* be the topological dual of H. Then: "For every $\varphi \in H^*$, there exists a unique element $y_\varphi \in H$ such that for all $x \in H$, we have:*

$$\varphi(x) = \langle x, y_\varphi\rangle.$$

Moreover,

$$\|\varphi\|_{H^*} = \|y_\varphi\|_H.$$

Example 3.13 *Let $H = l^2(\mathbb{N}) = \{(x_n);\quad \sum_{n\geq 0} x_n^2 < \infty\}$ equipped with the inner product*

$$\langle x, y\rangle = \sum_{n\geq 0} x_n y_n.$$

Let $p \in \mathbb{N}$ be fixed, and consider the application

$$\begin{aligned} \varphi : H &\to \mathbb{R} \\ x &\to \varphi(x) = x_p \end{aligned}$$

φ is linear (obviously) and continuous. Indeed, let $x \in H = l^2(\mathbb{N})$, then $x = (x_n)_{n\in\mathbb{N}}$, and we have:

$$|\varphi(x)| = |x_p| = \sqrt{x_p^2} \leq \sqrt{\sum_{n\geq 0} x_n^2} = \|x\|.$$

so that $\varphi \in H^*$.

We look for the unique element $y_\varphi \in H$ *such that* $\varphi(x) = \langle y_\varphi, x\rangle$. *Let us take*

$$y_\varphi = (0, ..., 1, 0, ...) = e_p.$$

We can easily see that $y_\varphi \in l^2(\mathbb{N})$ *and*

$$\langle y_\varphi, x\rangle = x_p = \varphi(x).$$

Proof of the theorem. Let $\varphi \in H^*$.

1. If $\varphi = 0 \in H^*$, it suffices to take $y_\varphi = 0$.

2. Suppose that $\varphi \neq 0$. Then

$$F = \ker\varphi = \{x \in H : \varphi(x) = 0\}$$

is a closed subspace of H, and $\ker\varphi \neq H$. Moreover,

$$F^\perp \neq \{0\} \quad (F \neq H)$$

Let $z \in F^\perp$ such that $\varphi(z) = 1$. Then, for all $x \in H$, we have

$$x = x - \varphi(x)z + \varphi(x)z$$

Note that $x - \varphi(x)z \in \ker\varphi$, because

$$\varphi\Big(x - \varphi(x)z\Big) = \varphi(x) - \varphi(x)\varphi(z) = 0.$$

So we get

$$\begin{aligned} \langle x, z\rangle &= \langle x - \varphi(x)z + \varphi(x)z, z\rangle \\ &= \langle x - \varphi(x)z, z\rangle + \langle \varphi(x)z, z\rangle \\ &= \varphi(x)\|z\|^2 \end{aligned}$$

hence

$$\varphi(x) = \frac{1}{\|z\|^2}\langle x, z\rangle = \langle x, \frac{z}{\|z\|^2}\rangle;$$

it suffices to take $y_\varphi = \dfrac{z}{\|z\|^2} \in H$.

■

Remark 3.10 *Let* $(H, \langle\cdot,\cdot\rangle)$ *be a Hilbert space. Then the topological dual* H^* *is itself a Hilbert space with the following inner product:*

$$\textit{For all } y, z \in H: \quad \langle \varphi_y, \varphi_z\rangle^* = \langle z, y\rangle.$$

Note that this inner product on H^* *induces the already existing norm on* H^*.

In what follows, we will see an important consequence of the Riesz theorem concerning reflexivity in Hilbert spaces. First, we introduce some commonly used notions.

Let E be a Banach space, and let E^* be its dual equipped with the dual norm:

$$\|f\|_{E^*} = \sup_{\|x\|\leq 1} |f(x)|.$$

Let $(E^*)^* = E^{**}$ be its bidual, i.e., the dual of E^*, equipped with the norm:

$$\|\xi\|_{E^{**}} = \sup_{\|f\|_{E^*}\leq 1} |\xi(f)|.$$

Definition 3.11 (Canonical injection of E into E^{**}) *A canonical injection $J : E \to E^{**}$ is defined as follows: for a fixed $x \in E$, the map $f \mapsto f(x)$ from E^* to $\mathbb{R}$ defines a linear continuous form on E^*, i.e., an element of E^{**}, denoted by Jx. So we have:*

$$\begin{aligned} J : E &\to E^{**} \\ x &\mapsto Jx, \end{aligned}$$

where:

$$\begin{aligned} Jx : E^* &\to \mathbb{R} \\ f &\mapsto Jx(f) = f(x). \end{aligned}$$

It is clear that J is linear and that J is an isometry, i.e., $\|Jx\|_{E^{**}} = \|x\|_E$ for all $x \in E$. Indeed:

$$\|Jx\|_{E^{**}} = \sup_{\|f\|_{E^*}\leq 1} |Jx(f)| = \sup_{\|f\|_{E^*}\leq 1} |f(x)| = \|x\|_E.$$

Definition 3.12 (Reflexive space) *Let E be a Banach space and let J be the canonical injection from E into E^{**}. We say that E is **reflexive** if $J(E) = E^{**}$.*

When E is reflexive, we implicitly identify E and E^{**} (via the isomorphism J).

Corollary 3.5 *Every Hilbert space is reflexive.*

Proof. Let H be a Hilbert space. According to Theorem 3.4, the application

$$\begin{aligned} \phi_H : H &\to H^* \\ y &\to \phi_H(y) = \varphi_y \end{aligned}$$

where $\varphi_y(x) = \langle x, y\rangle$, is semi-linear, bijective, and isometric. Let $J : H \to H^{**}$ be the canonical map and let $h \in H^{**}$, then the application

$$\begin{aligned} h \circ \phi_H : H &\to \mathbb{R} \\ y &\mapsto h \circ \phi_H(y) = h(\varphi_y) \end{aligned}$$

is a continuous linear form on H ($h \circ \phi_H \in H^*$), so there exists a unique element $x \in H$ such that for all $y \in H$, we have

$$h \circ \phi_H(y) = \langle x, y\rangle,$$

i.e.,

$$h(\varphi_y) = \langle x, y \rangle,$$

from which we get

$$h(\varphi_y) = \langle x, y \rangle = \varphi_y(x) = J(x)(\varphi_y),$$

hence,

$$h = J(x).$$

In other words, J is surjective, thus H is reflexive. ■

3.9 Bessel's Inequality, Parseval's Identity, and Hilbert Basis

Let $(E, \|\cdot\|)$ be a pre-Hilbert space of infinite dimension.

Theorem 3.5 (Bessel's Inequality) *Let $\{e_n\}_{n\in\mathbb{N}^*}$ be an orthonormal family in E. Then for every $x \in E$ we have:*

$$\sum_{i=1}^{n} |\langle e_i, x\rangle|^2 \leq \|x\|^2. \tag{3.12}$$

Proof. Let $F = \text{Vect}\{e_1, ..., e_n\}$ be a finite-dimensional subspace of E. Then the orthogonal projection onto F is well-defined. Since the family $\{e_1, ..., e_n\}$ is orthonormal, according to Proposition 3.10, the projection is written for any $x \in E$ as:

$$P_F(x) = \sum_{i=1}^{n} \langle x, e_i\rangle e_i,$$

hence,

$$\begin{aligned}
\|P_F(x)\|^2 &= \left\|\sum_{i=1}^{n} \langle x, e_i\rangle e_i\right\|^2 \\
&= \left\langle \sum_{i=1}^{n} \langle x, e_i\rangle e_i, \sum_{i=1}^{n} \langle x, e_i\rangle e_i \right\rangle \\
&= \sum_{i=1}^{n} |\langle x, e_i\rangle|^2.
\end{aligned}$$

We conclude using Corollary 3.4 which states that $\|P_F(x)\| \leq \|x\|$, thus the result follows. ■

Corollary 3.6 *Let $\{e_n\}_{n\in\mathbb{N}^*}$ be an orthonormal family in E. Then for every $x \in E$, the series with general term $|\langle x, e_n\rangle|^2$ is convergent, moreover:*

$$\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 \leq \|x\|^2. \tag{3.13}$$

Proof. The sequence of partial sums is increasing (since the general term is positive) and bounded (by inequality 3.12), hence the series is convergent. The upper bound (3.13) follows immediately by taking the limit. ■

Theorem 3.6 (Parseval's Identity) *Let $x \in E$. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal family in E. Then, for every $x \in \overline{Vect\{e_n\}_{n\in\mathbb{N}}}$, we have the identity known as* ***Parseval's Identity****:*

$$\sum_{n=0}^{\infty} |\langle x, e_n\rangle|^2 = \|x\|^2. \tag{3.14}$$

Proof. Let $F = Vect\{e_n\}_{n\in\mathbb{N}}$, and fix $N \in \mathbb{N}$. Define $F_N = Vect\{e_0, ..., e_N\}$. F_N is a finite-dimensional subspace of F, so the orthogonal projection onto F_N, denoted by P_{F_N}, exists. Since the family $\{e_n\}_{n\in\mathbb{N}}$ is orthonormal, then for all $x \in E$ we have $P_{F_N}(x) = \sum_{n=0}^{N}\langle x, e_n\rangle e_n$, and therefore:

$$\|P_{F_N}(x)\|^2 = \sum_{n=0}^{N} |\langle x, e_n\rangle|^2.$$

Let $x \in \overline{F}$. By definition, for every $\varepsilon > 0$, there exists a finite subset J of $\mathbb{N}$ and a family of scalars $\{\alpha_j\}_{j\in J}$ such that:

$$\|x - \sum_{j\in J}\alpha_j e_j\|^2 \leq \varepsilon \tag{3.15}$$

Let $N \geq \max J$. Extend the family $\{\alpha_j\}_{j\in J}$ by zeros to the family $\{\alpha_n\}_{n\in\{0,\ldots,N\}}$ (note that $\{\alpha_j\}_{j\in J} \subset \{\alpha_n\}_{n\in\{0,\ldots,N\}}$). Then, on one hand, by Bessel's inequality:

$$\|x - P_{F_N}(x)\|^2 = \|x\|^2 - \|P_{F_N}(x)\|^2 = \|x\|^2 - \sum_{n=0}^{N} |\langle x, e_n\rangle|^2 \geq 0, \tag{3.16}$$

On the other hand, since $\sum_{n=0}^{N}\alpha_n e_n \in F_N$ and from (3.15) we have:

$$\|x - P_{F_N}(x)\|^2 = d(x, F_N) \leq \|x - \sum_{n=0}^{N}\alpha_n e_n\|^2 = \|x - \sum_{j\in J}\alpha_j e_j\|^2 \leq \varepsilon. \tag{3.17}$$

Combining (3.16) and (3.17), we get:

$$\|x\|^2 - \varepsilon \leq \|P_{F_N}(x)\|^2 = \sum_{n=0}^{N} |\langle x, e_n\rangle|^2 \leq \|x\|^2,$$

Since the series is convergent by Corollary 3.6, taking the limit as $N \to +\infty$ gives:

$$\|x\|^2 - \varepsilon \leq \sum_{n=0}^{+\infty} |\langle x, e_n\rangle|^2 \leq \|x\|^2,$$

Letting $\varepsilon \to 0$, we obtain $\displaystyle\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 = \|x\|^2$. ∎

Definition 3.13 (Total Family) *A family $\{e_n\}_{n\in I\subset\mathbb{N}}$ of vectors in E is said to be* ***total*** *if the vector subspace it generates is dense in E, i.e.,*

$$E = \overline{Vect\{\{e_n\}_{n\in I}\}}.$$

Definition 3.14 (Hilbert Basis) *Let E be a pre-Hilbert space. A* ***Hilbert basis*** *of E is any total orthonormal family.*

In other words, a family $\{e_n\}_{n\in I}$ is a Hilbert basis of E if:

1. $\langle e_n, e_m\rangle = \begin{cases} 1, & \text{if } n = m \\ 0, & \text{if } n \neq m. \end{cases}$

2. $E = \overline{Vect\{\{e_n\}_{n\in I}\}}$.

Remark 3.11 *From definition 3.14, for all $x \in E$, there exists a family of scalars $(\alpha_i)_{i\in I} \subset \mathbb{K}$ such that*

$$x = \sum_{i\in I} \alpha_i x_i.$$

Example 3.14 1. *Consider the Hilbert space ℓ^2. Define the family $\{e_n\}_{n\in\mathbb{N}}$ by:*

$$e_1 = (1, 0, ...), \quad e_2 = (0, 1, 0, ...), \quad ..., \quad e_n = (0, ...0, \underbrace{1}_{\text{position } n}, 0, ...), ...$$

Then $\{e_n\}_{n\in\mathbb{N}}$ is a Hilbert basis of ℓ^2. Clearly, it is an orthonormal family in ℓ^2, and we now show that $\ell^2 = \overline{Vect\{\{e_n\}_{n\in\mathbb{N}}\}}$.
Let $x = (x_n)_n \in \ell^2$, since $\sum_{n=0}^{+\infty} |x_n|^2 < \infty$, then for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that:

$$\sum_{n=N+1}^{+\infty} |x_n|^2 < \varepsilon^2.$$

Define

$$y_N = \sum_{n=0}^{+\infty} x_n e_n,$$

then $y_N \in Vect\{\{e_n\}_{n\in\mathbb{N}}\}$ and we have:

$$\|x - y_N\|_2 = \left(\sum_{n=N+1}^{+\infty} |x_n|^2\right)^{\frac{1}{2}} < \varepsilon.$$

Thus, $Vect\{\{e_n\}_{n\in\mathbb{N}}\}$ is dense in ℓ^2, and therefore $\{e_n\}_{n\in\mathbb{N}}$ is a Hilbert basis of ℓ^2.

2. *Consider the space $E = C([-1, 1], \mathbb{R})$ equipped with the inner product:*

$$\langle f, g\rangle = \int_0^T f(t)g(t)\, dt,$$

Let $B = \{x^n, n \in \mathbb{N}\}$ be the family of polynomials. Recall the Weierstrass approximation theorem:

Theorem 3.7 *Every continuous function on an interval $[a, b]$ is the uniform limit of a sequence of polynomial functions.*

By this theorem, the family B is total in E with respect to the uniform norm $\|\cdot\|_\infty$. From the following inequality:

$$\|f\|^2 = \int_{-1}^{1} |f(t)|^2\, dt \leq 2 \sup_{t\in[-1,1]} |f(t)|^2 = 2\|f\|_\infty^2,$$

the family B remains total in E when equipped with the norm $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$. The Gram-Schmidt process allows us to construct an orthonormal family, and hence a Hilbert basis.

3. *Let $E = C_T(\mathbb{R},\mathbb{R})$ be the vector space of continuous, T-periodic functions on $\mathbb{R}$, equipped with the inner product:*
$$\langle f, g\rangle = \int_0^T f(t)g(t)\,dt.$$
Let $w = \frac{2\pi}{T}$. The sequence of functions:
$$\frac{1}{\sqrt{T}},\quad \sqrt{\frac{2}{T}}\cos(wt),\quad \sqrt{\frac{2}{T}}\sin(wt),\quad ...,\quad \sqrt{\frac{2}{T}}\cos(nwt),\quad \sqrt{\frac{2}{T}}\sin(nwt), ...$$
forms a Hilbert basis for E. (This is a consequence of the Weierstrass approximation theorem in its trigonometric form.)

Remark 3.12 *When E is infinite-dimensional, the concept of a Hilbert basis is a topological notion, not an algebraic or vectorial one; because the notion of "infinite sum" has no meaning in linear algebra.*

Theorem 3.8 *Let $(E, \|\cdot\|)$ be a (real) pre-Hilbert space and let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal family in E. The following statements are equivalent:*

(i) *$\{e_n\}_{n\in\mathbb{N}}$ is a Hilbert basis of E.*

(ii) *(Parseval's Identity): for all $x \in E$, we have*
$$\sum_{n=0}^{\infty} |\langle x, e_n\rangle|^2 = \|x\|^2.$$

(iii) *For all $x \in E$, the series with general term $\langle x, e_n\rangle e_n$ converges in E and we have:*
$$x = \sum_{n=0}^{\infty}\langle x, e_n\rangle e_n. \tag{3.18}$$

(iv) *For all $x, y \in E$, the numerical series with general term $\langle x, e_n\rangle\langle y, e_n\rangle$ converges in $\mathbb{R}$ and we have:*
$$\langle x, y\rangle = \sum_{n=0}^{\infty}\langle x, e_n\rangle\langle y, e_n\rangle. \tag{3.19}$$

Proof.

1. The implication $(i) \Rightarrow (ii)$ is immediate via Theorem 3.6.

2. We prove the implication $(ii) \Rightarrow (iii)$. Let $x \in E$, suppose that:
$$\sum_{n=0}^{\infty} |\langle x, e_n\rangle|^2 = \|x\|^2.$$
Let F be the vector subspace generated by the family $\{e_n\}_{n\in\mathbb{N}}$. For every $N \in \mathbb{N}$, let $F_N = \text{span}\{e_0, ..., e_N\}$. F_N is a finite-dimensional subspace of F, so the orthogonal projection onto F_N, denoted P_{F_N}, exists. Hence, for all $x \in E$, we have $P_{F_N}(x) = \sum_{n=0}^{N}\langle x, e_n\rangle e_n$ and thus
$$\|P_{F_N}(x)\|^2 = \sum_{n=0}^{N} |\langle x, e_n\rangle|^2.$$

Therefore, we get:

$$\begin{aligned}\lim_{N\to+\infty}\|x-\sum_{n=0}^{N}\langle x,e_n\rangle e_n\|^2 &= \lim_{N\to+\infty}\|x-P_{F_N}(x)\|^2\\ &= \lim_{N\to+\infty}(\|x\|^2-\|P_{F_N}(x)\|^2)\\ &= \|x\|^2-\lim_{N\to+\infty}\sum_{n=0}^{N}|\langle x,e_n\rangle|^2\\ &= \|x\|^2-\sum_{n=0}^{\infty}|\langle x,e_n\rangle|^2=0\quad\text{by hypothesis.}\end{aligned}$$

Hence,

$$\lim_{N\to+\infty}\|x-\sum_{n=0}^{N}\langle x,e_n\rangle e_n\|^2=0. \tag{3.20}$$

By the continuity of the norm map $x\to\|x\|$, we have

$$\lim_{N\to+\infty}\|x-\sum_{n=0}^{N}\langle x,e_n\rangle e_n\|^2=\|x-\sum_{n=0}^{+\infty}\langle x,e_n\rangle e_n\|^2. \tag{3.21}$$

From relations (3.20) and (3.21), we deduce:

$$\|x-\sum_{n=0}^{+\infty}\langle x,e_n\rangle e_n\|^2=0,$$

which implies that $x=\sum_{n=0}^{+\infty}\langle x,e_n\rangle e_n$ in the sense of norm convergence.

3. The implication $(iii)\Rightarrow(i)$ is trivial. Therefore, $(i)\Leftrightarrow(ii)\Leftrightarrow(iii)$.

4. We show the equivalence $(ii)\Leftrightarrow(iv)$. For the implication $(iv)\Rightarrow(ii)$, it suffices to take $x=y$ in the identity (3.19) to obtain Parseval's identity. Now we prove the implication $(ii)\Rightarrow(iv)$. Let $x,y\in E$, and consider the polarization identity:

$$\langle x,y\rangle=\frac{1}{4}(\|x+y\|^2-\|x-y\|^2).$$

Substituting $\langle x,e_n\rangle e_n$, $\langle y,e_n\rangle e_n$ into this equality, we get

$$\langle x,e_n\rangle\langle y,e_n\rangle=\frac{1}{4}(|\langle e_n,x+y\rangle|^2-|\langle e_n,x-y\rangle|^2),$$

According to Parseval's identity applied to $x+y$ and $x-y$, we have

$$\sum_{n=0}^{+\infty}|\langle x+y,e_n\rangle e_n|^2=\|x+y\|^2<\infty\quad\text{and}\quad\sum_{n=0}^{+\infty}|\langle x-y,e_n\rangle e_n|^2=\|x-y\|^2<\infty.$$

So the series with general term $\langle x,e_n\rangle\langle y,e_n\rangle$ is absolutely convergent, and using the polarization identity we deduce:

$$\begin{aligned}\langle x,y\rangle &= \frac{1}{4}\Big(\sum_{n=0}^{+\infty}|\langle x+y,e_n\rangle e_n|^2-\sum_{n=0}^{+\infty}|\langle x-y,e_n\rangle e_n|^2\Big)\\ &= \sum_{n=0}^{+\infty}\frac{1}{4}\Big(|\langle x+y,e_n\rangle e_n|^2-|\langle x-y,e_n\rangle e_n|^2\Big)\\ &= \sum_{n=0}^{+\infty}\langle x,e_n\rangle\langle y,e_n\rangle.\end{aligned}$$