We have:

$$f(x_n, y_n) - f(x_0, y_0) = f(x_n - x_0, y_n) + f(x_0, y_n - y_0),$$

so

$$\begin{aligned} \|f(x_n, y_n) - f(x_0, y_0)\|_F &\leq \|f(x_n - x_0, y_n)\|_F + \|f(x_0, y_n - y_0)\|_F \\ &\leq C\Big(\|x_n - x_0\|_1\|y_n\|_2 + \|x_0\|_1\|y_n - y_0\|_2\Big), \end{aligned}$$

Writing $y_n = (y_n - y_0) + y_0$, we obtain:

$$\|f(x_n, y_n) - f(x_0, y_0)\|_F \leq C\Big(\|x_n - x_0\|_1\|y_n - y_0\|_2 + \|x_n - x_0\|_1\|y_0\|_2 + \|x_0\|_1\|y_n - y_0\|_2\Big)$$

From relation (2.5), it follows that $\|f(x_n, y_n) - f(x_0, y_0)\|_F \to 0$, and thus f is continuous at (x_0, y_0).

■

Remark 2.2 *Theorem 2.2 can be generalized to the case of a multilinear map.*

2.5 Exercises

♦ On the Continuity of Linear Applications

Exercice 2.3 *Let $E = C([0,1], \mathbb{R})$. We define the application*

$$\begin{aligned} T : E &\to \mathbb{R} \\ f &\to T(f) = f(1). \end{aligned}$$

1. *Equip E with the norm $\|f\|_1 = \int_0^1 |f(x)|\, dx$. Let (f_n) be the sequence defined by $f_n(x) = \sqrt{n}x^n$.*
   (a) *Compute $\|f_n\|_1$.*
   (b) *Show that T is not continuous.*
2. *Now equip E with the norm $\|f\|_\infty = \sup\limits_{x\in[0,1]} |f(x)|$. Is the application T continuous?*

Exercice 2.4 *Determine whether the linear application $T : (E, N_1) \to (F, N_2)$ is continuous in the following cases:*

1. *$E = C([0,1], \mathbb{R})$ with the norm $\|f\|_1 = \displaystyle\int_0^1 |f(x)|\, dx$ and $T : (E, \|\cdot\|_1) \to (E, \|\cdot\|_1)$ defined by $T(f) = fg$, where $g \in E$ is fixed.*
2. *$E = \mathbb{R}_n[X]$ with the norm $\|P\| = \sum\limits_{k=0}^{n} |a_k|$ where $P(X) = \sum\limits_{k=0}^{n} a_k X^k$, and $T : (E, \|\cdot\|) \to (E, \|\cdot\|)$ defined by $T(P) = P'$.*

3. $E = \mathbb{R}[X]$ *with the norm* $\|P\| = \sum_{k=0}^{\infty} k!|a_k|$ *where* $P(X) = \sum_{k=0}^{\infty} a_k X^k$*, and* $T : (E, \|\cdot\|) \to (E, \|\cdot\|)$ *defined by* $T(P) = P'$.

4. $E = C([0,1], \mathbb{R})$ *with the norm* $\|f\|_2 = \left(\int_0^1 |f(x)|^2 \, dx \right)^{\frac{1}{2}}$*,* $F = C([0,1], \mathbb{R})$ *with the norm* $\|f\|_1 = \int_0^1 |f(x)| \, dx$*, and* $T : (E, \|\cdot\|_2) \to (E, \|\cdot\|_1)$ *defined by* $T(f) = fg$ *where* $g \in E$ *is fixed.*

♦ Norm Calculations in $\mathcal{L}(E, F)$

Exercice 2.5 *Compute the norm of the identity of* $(C^1([0,1], \mathbb{R}), \|\cdot\|)$ *in* $(C^1([0,1], \mathbb{R}), \|\cdot\|_1)$ *and of* $(C^1([0,1], \mathbb{R}), \|\cdot\|_1)$ *in* $(C^1([0,1], \mathbb{R}), \|\cdot\|)$*; where* $\|\cdot\|$ *and* $\|\cdot\|_1$ *are defined as follows:*

$$\|f\| = \|f\|_\infty + \|f'\|_\infty \, , \qquad \|f\|_1 = \|f'\|_\infty + |f(0)|.$$

Exercice 2.6 *Let* $E = C([0,1], \mathbb{R})$ *and* $F = C^1([0,1], \mathbb{R})$*. Equip these spaces with:*

$$\|f\|_E = \|f\|_\infty \quad \textit{and} \quad \|f\|_F = \|f\|_\infty + \|f'\|_\infty.$$

Define the application

$$\begin{array}{rcl} T : E & \to & F \\ f & \to & T(f) \end{array}$$

where $T(f) : [0,1] \to \mathbb{R}$ *is defined by* $T(f)(x) = \int_0^x f(t) \, dt$.

Show that the linear application T *is continuous and compute* $\|T\|_{\mathcal{L}(E,F)} \equiv \|T\|$.

Exercice 2.7 *Let* $E = C([0,1], \mathbb{R})$ *equipped with a norm* $\|\cdot\|$*. Study the continuity of the linear form* $\varphi : E \to \mathbb{R}$ *and compute its norm in the following cases:*

1. $\|f\| = \|f\|_\infty$ *and* $\varphi(f) = \int_0^1 f(t) \, dt$.

2. $\|f\| = \|f\|_\infty$ *and* $\varphi(f) = f(1) - f(0)$.

3. $\|f\| = \|f\|_1 = \int_0^1 |f(t)| \, dt$ *and* $\varphi(f) = \int_0^1 t f(t) \, dt$.

Exercice 2.8 *Let* $E = C([0,1], \mathbb{R})$ *and* $u \in L(E, E)$ *that maps* $f \in E$ *to the function* $u(f) \in E$ *defined by:*

$$u(f)(x) = f(x) - f(0), \quad \forall x \in [0,1].$$

1. *Show that for* E *equipped with the norm* $\|\cdot\|_\infty$*, the application* u *is continuous and compute its norm.*

2. *Show that for* E *equipped with the norm* $\|\cdot\|_1$*,* u *is not continuous.*

Exercice 2.9 *Let $E = C([a,b],\mathbb{R})$ equipped with the uniform convergence norm $\|\cdot\|_\infty$. Let p be an integrable function on $[a,b]$, and for each $f \in E$, consider the application:*

$$\begin{array}{rcl} \phi : E & \to & E \\ f & \to & \phi(f); \end{array}$$

where $\phi(f)(x) = \int_a^x f(t)p(t)\,dt, \quad$ for every $x \in [a,b]$.

1. *Show that the application $\phi : E \to E$ is linear and continuous.*

2. *Verify that*
$$\|\phi\|_{\mathcal{L}(E)} \leq \int_a^b |p(t)|\,dt.$$

3. *Suppose $p \geq 0$. Using a simple choice of f, show that:*
$$\|\phi\|_{\mathcal{L}(E)} = \int_a^b p(t)\,dt.$$

4. *Now suppose that p is continuous on $[a,b]$ and takes any sign. Let*
$$f_n(t) = \frac{np(t)}{\sqrt{1+n^2\big(p(t)\big)^2}}.$$

   (a) *Verify that $f_n \in E$ and compute $\|\phi(f_n)\|_\infty$.*

   (b) *Deduce that $\|\phi\|_{\mathcal{L}(E)} = \int_a^b |p(t)|\,dt$.*

♦ Continuous Linear Forms and Closed Hyperplanes

Exercice 2.10 *Let $E = C([0,1],\mathbb{R})$ equipped with the uniform norm $\|\cdot\|_\infty$. Define:*

$$F = \{f \in E; f(0) = 0 \ \text{ and } \ \int_0^1 f(t)\,dt \geq 1\}$$

Show that F is a closed subset of E.

Exercice 2.11 *Let $E = \ell^1$, the vector space of complex sequences $(x_n)_{n\in\mathbb{N}^*}$ such that $\sum\limits_{n=1}^{\infty} |x_n|$ converges. Equip it with the norm:*

$$\|x\|_E = \sum_{n=1}^{\infty} |x_n|.$$

Define:

$$F = \{x = (x_n)_{n\in\mathbb{N}^*} \in E : \sum_{n=1}^{\infty} |x_n| = 1\}$$

1. *Show that F is closed.*

2. *Is F bounded?*

Exercice 2.12 *Let E be a normed vector space and $u \in \mathcal{L}(E)$. Show that u is continuous if and only if $\{x \in E; \|u(x)\| = 1\}$ is closed.*

Exercice 2.13 *Let E be a normed $\mathbb{R}$-vector space and let $\Phi : E \to \mathbb{R}$ be a nonzero linear form. The goal is to show that Φ is continuous if and only if the kernel of Φ is closed.*

1. *Prove the direct implication.*
2. *Conversely, assume that the kernel of Φ, denoted by H, is closed. Fix $y \in E$ such that $\Phi(y) = 1$.*
   (a) *Show that $\Phi^{-1}\Big(\{1\}\Big)$ is closed.*
   (b) *Deduce that there exists $r > 0$ such that $B(0, r) \cap \Phi^{-1}\Big(\{1\}\Big) = \varnothing$*
   (c) *Show that: $x \in B(0, r) \Rightarrow |\Phi(x)| \leq 1$*
   (d) *Conclude.*

Exercice 2.14 *Let $(E, \|\cdot\|)$ be a normed $\mathbb{K}$-space and let $\varphi : E \to \mathbb{K}$ be a nonzero continuous linear form. Let $H = \ker \varphi$.*

1. *Show that*
$$\frac{|\varphi(x)|}{\|\varphi\|_{E'}} \leq d(x, H).$$
2. *Let $b \in E \setminus H$, and $x \in E$. Noting that $x = \frac{\varphi(x)}{\varphi(b)} b + x - \frac{\varphi(x)}{\varphi(b)} b$, show that there exist sequences $(x_n) \subset E \setminus H$, $(t_n) \subset \mathbb{R}$ and $(h_n) \subset H$ such that $\|x_n\| = 1$ and*
$$x = h_n + t_n x_n.$$
3. *Show that for all $x \in E$, we have*
$$d(x, H) = \frac{|\varphi(x)|}{\|\varphi\|_{E'}}.$$

Exercice 2.15 *Let ℓ_0 be the space of real sequences with only finitely many non-zero terms, equipped with the norm*
$$\|x\| = \sup_{n \in \mathbb{N}} |x_n|,$$
where $x = (x_n)_n \in \ell_0$. Consider the application
$$\begin{array}{rcl} \phi : \ell_0 & \to & \mathbb{R} \\ x & \mapsto & \displaystyle\sum_{n=0}^{\infty} \frac{x_n}{2^n}. \end{array}$$

1. *Show that ϕ is a nonzero continuous linear form and compute its norm.*
2. *Let $H = \ker \phi$ and $x \in \ell_0 \setminus H$. Show that there is no point h such that*
$$d(x, H) = d(x, h) = \|x - h\|.$$