Chapter 2

Continuous Linear Maps

2.1 Definitions and Properties

Definition 2.1 *Let E and F be two vector spaces over the same field $\mathbb{K}$. A map $f : E \to F$ is said to be **linear** from E to F if it satisfies, for all $\alpha, \beta \in \mathbb{K}$ and all $x, y \in E$:*

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y).$$

Example 2.1 *The following maps are linear:*

i)

$$\begin{array}{rcl} F : \mathbb{R}^n & \to & \mathbb{R} \\ x = (x_1, ..., x_n) & \to & \sum_{i=1}^{n} \lambda_i x_i, \quad \lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{R}^n. \end{array}$$

ii)

$$\begin{array}{rcl} T : C([0,1], \mathbb{R}) & \to & \mathbb{R} \\ f & \to & T(f) = \int_0^1 f(x)\, dx. \end{array}$$

iii)

$$\begin{array}{rcl} D : C^1([0,1], \mathbb{R}) & \to & C([0,1], \mathbb{R}) \\ f & \to & D(f) = f'. \end{array}$$

We denote by $L(E, F)$ the set of linear maps from E to F. This set forms a vector space over $\mathbb{K}$ when equipped with the following standard operations:

$$\begin{array}{rcl} L(E,F) \times L(E,F) & = & L(E,F) \\ (f,g) & = & f+g, \end{array}$$

$$\begin{array}{rcl} \mathbb{K} \times L(E,F) & = & L(E,F) \\ (\lambda, f) & = & \lambda f. \end{array}$$

Definition 2.2 (Linear Form) *A **linear form** on a $\mathbb{K}$-vector space E is any linear map from E to $\mathbb{K}$.*

Definition 2.3 (Algebraic Dual) *The set of all linear forms on E is denoted by:*

$$E^* = L(E, \mathbb{K}),$$

*and is called the **algebraic dual** of E.*

- From Example 2.1, we observe that $F \in (\mathbb{R}^n)^*$, $T \in (C([0,1], \mathbb{R}))^*$, but D is not a linear form.
- It is easy to see that E^* has a structure of a $\mathbb{K}$-vector space.

Definition 2.4 (Semi-linear Form) *If E is a $\mathbb{C}$-vector space, then a **semi-linear form** on E is any map $f : E \to \mathbb{C}$ such that:*

$$f(\alpha x + \beta y) = \overline{\alpha} f(x) + \overline{\beta} f(y), \quad \forall \alpha, \beta \in \mathbb{C}, \forall x, y \in E,$$

where $\overline{\alpha}$ and $\overline{\beta}$ denote the complex conjugates of α and β respectively.

The following proposition provides an important characterization of the continuity of a linear map.

Proposition 2.1 *Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two normed $\mathbb{K}$-vector spaces. A linear map $f : E \to F$ is continuous if and only if there exists a constant $C > 0$ such that:*

$$\forall x \in E, \quad \|f(x)\|_F \leq C\|x\|_E. \tag{2.1}$$

Proof.

($\Rightarrow$) Assume f is continuous on E, then in particular it is continuous at 0, so by definition:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x, \|x\|_E \leq \delta \Rightarrow \|f(x)\|_F \leq \varepsilon.$$

Taking $\varepsilon = 1$, for all $x \in E \setminus \{0\}$, let $y = \frac{\delta}{\|x\|_E} x$. Since $\|y\|_E = \delta$, then:

$$\|f(y)\|_F = \frac{\delta}{\|x\|_E} \|f(x)\|_F \leq 1,$$

hence:

$$\|f(x)\|_F \leq \frac{1}{\delta} \|x\|_E.$$

So we can take $C = \frac{1}{\delta}$.

($\Leftarrow$) If inequality 2.1 is satisfied, then for all $x, y \in E$, we have:

$$\|f(x) - f(y)\|_F = \|f(x - y)\|_F \leq C\|x - y\|_E,$$

so f is Lipschitz continuous, hence uniformly continuous, and therefore continuous on E.

■

Example 2.2 *1. Let $E = \mathbb{R}^N$ with the Euclidean norm $\|x\|_2 = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$. The linear map:*

$$\begin{array}{rcl} f : E & \to & \mathbb{R} \\ x & \mapsto & \displaystyle\sum_{i=1}^n x_i \end{array}$$

is continuous. Indeed, by the Cauchy-Schwarz inequality:

$$|f(x)| = |\sum_{i=1}^n x_i| \le \left(\sum_{i=1}^n 1^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}} = \sqrt{n}\|x\|_2.$$

So take $C = \sqrt{n}$.

2. Let $E = C([0,1], \mathbb{R})$ with the sup norm $\|\cdot\|_\infty$. The linear map:

$$\begin{array}{rcl} T : E & \to & \mathbb{R} \\ f & \mapsto & \displaystyle\int_0^1 f(x)\, dx \end{array}$$

is continuous because:

$$|T(f)| = \left|\int_0^1 f(x)dx\right| \le \int_0^1 |f(x)|dx \le \sup_{x\in[0,1]} |f(x)| \int_0^1 dx = \|f\|_\infty.$$

So take $C = 1$.

3. Let $E = C([0,1], \mathbb{R})$ and $F = C^1([0,1], \mathbb{R})$ with sup norms. Define:

$$\begin{array}{rcl} L : E & \to & F \\ f & \mapsto & L(f), \text{ where } L(f)(x) = \displaystyle\int_0^x f(t)dt. \end{array}$$

Then L is continuous because:

$$\|L(f)\|_\infty = \sup_{x\in[0,1]} \left|\int_0^x f(t)dt\right| \le \sup_{x\in[0,1]} \left(x \cdot \sup_{t\in[0,1]} |f(t)|\right) = \|f\|_\infty.$$

So take $C = 1$.

4. Consider $C([0,1], \mathbb{R})$ with the L^1-norm $\|f\|_1 = \int_0^1 |f(x)|dx$. Then the linear form:

$$\begin{array}{rcl} S : E & \to & \mathbb{R} \\ f & \mapsto & f(0) \end{array}$$

is not continuous. To see this, define a sequence (f_n) in E such that:

$$\lim_{n\to+\infty} \frac{|S(f_n)|}{\|f_n\|_1} = +\infty.$$

For example:

$$f_n(x) = \begin{cases} 1 - (n+1)x & \text{if } 0 \le x \le \frac{1}{n+1}, \\ 0 & \text{if } \frac{1}{n+1} < x \le 1. \end{cases}$$

Then $f_n(0) = 1$ and $\|f_n\|_1 = \frac{1}{n+1}$, so:

$$\lim_{n\to+\infty} \frac{1}{\|f_n\|_1} = +\infty.$$

As a direct consequence of Proposition 2.1, we have:

Corollary 2.1 *Let $f : E \to F$ be a linear map. The following statements are equivalent:*

1. *f is continuous on E.*
2. *f is continuous at 0_E.*
3. *f is uniformly continuous on E.*

Notation: We denote by $\mathcal{L}(E, F)$ the space of continuous linear maps from E to F. If $E = F$, we write $\mathcal{L}(E, E) = \mathcal{L}(E)$.

Definition 2.5 (Topological Dual) *The **topological dual** of E is the space of continuous linear forms on E, denoted by:*

$$E' = \mathcal{L}(E, \mathbb{K}).$$

Example 2.3 *From Example 2.2 and using associated norms, we see that $f \in (\mathbb{R}^n)'$, $T \in (C([0,1], \mathbb{R}))'$, $L \in \mathcal{L}\left(C([0,1], \mathbb{R}), C^1([0,1], \mathbb{R})\right)$ but $S \notin (C([0,1], \mathbb{R}))'$.*

Proposition 2.2 *Let $(E, \|\cdot\|)$ be a normed space of finite dimension. Then for any normed space F:*

$$L(E, F) = \mathcal{L}(E, F).$$

That is, every linear map from E to F is continuous.

In particular, we have $E^* = E'$.

Proof. Fix a basis $\{e_i\}_{1 \leq i \leq n}$ of E, and equip E with the norm:

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \quad \forall x \in E,$$

where $(x_1, x_2, ..., x_n)$ are the coordinates of x in the chosen basis. Suppose f is a linear map from E to any normed space $(F, \|\cdot\|)$. Then:

$$\|f(x)\| = \left\| f\left(\sum_{i=1}^{n} x_i e_i\right) \right\| = \left\| \sum_{i=1}^{n} x_i f(e_i) \right\| \leq \sum_{i=1}^{n} |x_i| \|f(e_i)\|.$$

Let $C = \max_{1 \leq i \leq n} \|f(e_i)\|$, then:

$$\|f(x)\| \leq C\|x\|_1,$$

which shows that f is continuous. Moreover, taking $F = \mathbb{K}$ gives $E^* = E'$. ∎

2.2 Continuous Linear Forms and Closed Hyperplanes

Definition 2.6 *For any linear form f on a vector space E, the **hyperplane** is the vector subspace H of E defined by*

$$H = \ker f = \{x \in E : f(x) = 0\} = f^{-1}(\{0\}).$$

Proposition 2.3 *Let E be a normed space and f a linear form on E. Then the following assertions are equivalent:*

1. *f is continuous.*

2. *$H = \ker f$ is a closed hyperplane in E.*

Proof.

($\Rightarrow$) If f is continuous, then H is closed as the preimage of the closed set $\{0\}$ under the continuous map f.

($\Leftarrow$) Suppose that the hyperplane $H = \ker f$ is closed in E. Let $a \in E$ such that $f(a) = 1$ (if $f(a) \neq 1$, we can take $\frac{a}{f(a)}$, so that $f\left(\frac{a}{f(a)}\right) = 1$).

It is clear that the set $a + H$ defined by

$$a + H = \{a + x : x \in H\}$$

is closed in E since the map $x \mapsto a + x$ is a homeomorphism of E. Note that $0 \notin a + H$ because:

$$\forall x \in H : f(a + x) = f(a) + f(x) = f(a) = 1 \neq 0 \Rightarrow \forall x \in H : a + x \neq 0.$$

Therefore, there exists $r > 0$ such that $B(0, r) \cap (a + H) = \emptyset$. Let $x \in B(0, r)$.

If $|f(x)| > 1$, then

$$\left\| \frac{x}{f(x)} \right\| = \frac{\|x\|}{|f(x)|} < \|x\| < r.$$

Moreover, $f\left(\frac{x}{f(x)}\right) = 1 = f(a)$, which implies that $f\left(\frac{x}{f(x)} - a\right) = 0$, hence $\frac{x}{f(x)} - a \in H$, and so: $\frac{x}{f(x)} \in a + H$, which is a contradiction. Thus, $|f(x)| \leq 1$.

Let $x \in E$, $x \neq 0$, then

$$\left\| \frac{rx}{2\|x\|} \right\| = \frac{r}{2} < r,$$

so

$$\frac{r}{2\|x\|} |f(x)| = \left| f\left(\frac{rx}{2\|x\|}\right) \right| \leq 1,$$

and thus

$$|f(x)| \leq \frac{2}{r} \|x\|,$$

which shows that f is continuous.

■

2.3 Space $\mathcal{L}(E,F)$

Proposition 2.4 *Let $(E,\|\cdot\|_E)$, $(F,\|\cdot\|_F)$ be two normed vector spaces over $\mathbb{K}$. The space $\mathcal{L}(E,F)$ is a normed vector space with the norm:*

$$\|f\| = \sup_{x\in E\setminus\{0\}} \frac{\|f(x)\|_F}{\|x\|_E}.$$

Proof. It is easy to see that $\mathcal{L}(E,F)$ is a $\mathbb{K}$-vector subspace. We will show that $\|\cdot\| : \mathcal{L}(E,F) \to \mathbb{K}$ defines a norm; let $f\in\mathcal{L}(E,F)$.

1. Clearly, if $f=0$ then $\|f\|=0$. Conversely, if $\|f\|=0$, then for all $x\in E\setminus\{0\}$ we have $\frac{\|f(x)\|_F}{\|x\|_E}=0$, hence $\|f(x)\|_F=0$ and since $\|\cdot\|_F$ is a norm on F, we deduce $f(x)=0$ for all $x\in E\setminus\{0\}$. Also, $f(0)=0$, hence $f=0$.

2. Let $x\in E\setminus\{0\}$ and $\lambda\in\mathbb{K}$, we have:
$$\|\lambda f\| = \sup_{x\in E\setminus\{0\}} \frac{\|\lambda f(x)\|_F}{\|x\|_E} = |\lambda| \sup_{x\in E\setminus\{0\}} \frac{\|f(x)\|_F}{\|x\|_E} = |\lambda|\|f\|.$$

3. Let $f,g\in\mathcal{L}(E,F)$, then:
$$\|f+g\| = \sup_{x\in E\setminus\{0\}} \frac{\|f(x)+g(x)\|_F}{\|x\|_E},$$
Using that $\|\cdot\|_F$ is a norm on F and the subadditivity of the supremum, we get:
$$\|f+g\| \le \sup_{x\in E\setminus\{0\}} \frac{\|f(x)\|_F+\|g(x)\|_F}{\|x\|_E} \le \sup_{x\in E\setminus\{0\}} \frac{\|f(x)\|_F}{\|x\|_E} + \sup_{x\in E\setminus\{0\}} \frac{\|g(x)\|_F}{\|x\|_E}.$$
Hence, $\|f+g\|\le\|f\|+\|g\|$.

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Proposition 2.5 *Let $(E,\|\cdot\|_E)$, $(F,\|\cdot\|_F)$ be two normed vector spaces and f a continuous linear map. Then:*

$$\sup_{x\in E\setminus\{0\}} \frac{\|f(x)\|_F}{\|x\|_E} = \sup_{\|x\|_E=1} \|f(x)\|_F = \sup_{\|x\|_E\le 1} \|f(x)\|_F = \inf\Big\{c>0 : \|f(x)\|_F \le c\|x\|_E\Big\}.$$

Proof. Let $A = \sup_{x\in E\setminus\{0\}} \frac{\|f(x)\|_F}{\|x\|_E}$, $B=\sup_{\|x\|_E=1}\|f(x)\|_F$, $C=\sup_{\|x\|_E\le1}\|f(x)\|_F$, $D=\inf\Big\{\lambda>0 : \|f(x)\|_F\le\lambda\|x\|_E\Big\}$.

(i) For all $x\in E\setminus\{0\}$, we have:

$$A = \sup_{x\in E\setminus\{0\}} \frac{\|f(x)\|_F}{\|x\|_E} = \sup_{x\in E\setminus\{0\}} \left\|\frac{1}{\|x\|_E} f(x)\right\|_F = \sup_{x\in E\setminus\{0\}} \left\|f\Big(\frac{x}{\|x\|_E}\Big)\right\|_F,$$

Let $y=\dfrac{x}{\|x\|_E}$, so:

$$A = \sup_{\|y\|_E=1} \Big\|f(y)\Big\|_F = B.$$

(ii) Since $\Big\{\|f(x)\|_F; \|x\|_E = 1\Big\} \subset \Big\{\|f(x)\|_F; \|x\|_E \leq 1\Big\}$, we deduce that $B \leq C$.

(iii) We verify that C is a lower bound for the set $\Big\{\lambda > 0 : \|f(x)\|_F \leq \lambda\|x\|_E\Big\}$; indeed, for any $\lambda > 0$ such that $\|f(x)\|_F \leq \lambda\|x\|_E$, then for all $x \in E$ with $\|x\|_E \leq 1$, we have:

$$\|f(x)\|_F \leq \lambda\|x\|_E \leq \lambda,$$

hence $\sup_{\|x\|_E \leq 1} \|f(x)\|_F \leq \lambda$ and therefore $C \leq D$.

(iv) For all $x \in E \setminus \{0\}$, we have:

$$\frac{\|f(x)\|_F}{\|x\|_E} \leq A,$$

i.e. $\|f(x)\|_F \leq A\|x\|_E$, hence by the definition of D, we deduce $D \leq A$ since A belongs to the set $\Big\{\lambda > 0 : \|f(x)\|_F \leq \lambda\|x\|_E\Big\}$.

We conclude that $A = B \leq C \leq D \leq A$, which shows that all four quantities are equal. ■

Remark 2.1 *From the definition of $\|f\|$, we can write the following inequality:*

$$\forall x \in E : \quad \|f(x)\|_F \leq \|f\|\|x\|_E.$$

Example 2.4 *1. Let $E = C([0,1], \mathbb{R})$ with the supremum norm $\|\cdot\|_\infty$. We have already shown that the map:*

$$\begin{aligned} T : E &\to \mathbb{R} \\ f &\to T(f) = \int_0^1 f(x)\, dx. \end{aligned}$$

*belongs to $\mathcal{L}(E, \mathbb{R})$. Now we compute $\|T\| = \sup\limits_{\|f\|_\infty \leq 1} |T(f)|$. By the inequality (**??**) and the definition of $\|T\|$ (i.e. $\|T\| = D$), we have $\|T\| \leq 1$. On the other hand, let us look for an element in $\overline{B}(0,1)$ such that $|T(f_0)| \geq 1$; consider $f_0 \equiv 1$, then:*

$$\|f_0\|_\infty = \sup_{x \in [0,1]} |1| = 1, \quad \textit{and} \quad T(f_0) = \int_0^1 f_0(x)\, dx = 1$$

which implies $\|T\| \geq 1$. Therefore, $\|T\| = 1$.

2. Let $E = C([0,1], \mathbb{R})$ with the norm:

$$\|f\|_1 = \int_0^1 |f(x)|\, dx.$$

Consider the operator $\psi : E \to E$ defined by:

$$\psi(f)(x) = \int_0^x f(t)\, dt.$$

First, note that ψ is linear. To verify continuity, let $f \in E$; for all $x \in [0,1]$, we have:

$$|\psi(f)(x)| \leq \int_0^x |f(t)|\, dt \leq \int_0^1 |f(t)|\, dt = \|f\|_1,$$

so:

$$\|\psi(f)\|_1 = \int_0^1 |\psi(f)(x)|\,dx \le \int_0^1 \|f\|_1\,dx = \|f\|_1, \tag{2.2}$$

hence $\psi \in \mathcal{L}(E)$*. From (2.2), we get* $\|\psi\| \le 1$*.*

To prove the reverse inequality, consider the sequence (f_n) *defined by:*

$$f_n(x) = ne^{-nx}, \quad x \in [0,1].$$

We know:

$$\|\psi(f_n)\|_1 \le \|\psi\|\|f_n\|_1, \tag{2.3}$$

Moreover:

$$\begin{aligned} \|\psi(f_n)\|_1 &= \int_0^1 |\psi(f_n)(x)|\,dx = \int_0^1 \left|\int_0^x ne^{-nt}\,dt\right|\,dx = 1 - \frac{1-e^{-n}}{n}. \\ \|f_n\|_1 &= \int_0^1 ne^{-nx}\,dx = 1 - e^{-n}. \end{aligned}$$

Substituting into (2.3) gives:

$$\|\psi\| \ge \frac{\|\psi(f_n)\|_1}{\|f_n\|_1} = \frac{1}{1-e^{-n}} - \frac{1}{n},$$

Taking the limit yields $\|\psi\| \ge 1$*, hence* $\|\psi\| = 1$*.*

Theorem 2.1 *If* F *is a Banach space, then* $\mathcal{L}(E,F)$ *is also a Banach space.*

Proof. Assume (f_n) is a Cauchy sequence in $(\mathcal{L}(E,F), \|\cdot\|)$, i.e.:

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall p, q \in \mathbb{N} : p > q \ge n_0 \Rightarrow \|f_p - f_q\| \le \varepsilon.$$

Then for every $x \in E$:

$$\|f_p(x) - f_q(x)\|_F \le \|f_p - f_q\|\|x\|_E \le \varepsilon\|x\|_E. \tag{2.4}$$

So $(f_n(x))$ is a Cauchy sequence in F, which is complete, so it converges to a limit denoted $f(x)$: $\lim\limits_{n\to+\infty} f_n(x) = f(x)$. We verify that $f : E \to F$ is linear. For $x, y \in E$ and $\alpha \in \mathbb{K}$:

$$\begin{aligned} f(x+\alpha y) &= \lim_{n\to+\infty} f_n(x+\alpha y) \\ &= \lim_{n\to+\infty} f_n(x) + \alpha f_n(y) \\ &= \lim_{n\to+\infty} f_n(x) + \alpha \lim_{n\to+\infty} f_n(y) \\ &= f(x) + \alpha f(y). \end{aligned}$$

Since the norm function $F \ni x \mapsto \|x\|_F$ is continuous, for fixed $n \ge n_0$, we have:

$$\lim_{q\to+\infty} \|f_n(x) - f_q(x)\|_F = \|f_n(x) - f(x)\|_F,$$

From (2.4) we conclude:

$$\|(f_n - f)(x)\|_F = \|f_n(x) - f(x)\|_F \le \varepsilon\|x\|_E,$$

So $f_n - f$ is continuous, and since f_n is continuous, f is continuous as the difference of continuous functions. Hence $f \in \mathcal{L}(E,F)$, and:

$$\lim_{q\to+\infty} \|f_n - f_q\| = \|f_n - f\| \le \varepsilon, \quad \forall q \ge n_0.$$

Thus, (f_n) converges in $\mathcal{L}(E,F)$. ∎

2.4 Continuous Multilinear Applications

Definition 2.7 (Multilinear and Bilinear Applications) *Let $E_1, ..., E_n$ and F be $\mathbb{K}$-vector spaces. A map $f : E_1 \times ... \times E_n \to F$ is said to be **multilinear** from $E_1 \times ... \times E_n$ into F if for every $i \in \mathbb{N}$, $1 \leq i \leq n$, and every $y_j \in E_j$, with $i \neq j$, the map*

$$\begin{array}{rcl} E_i & \to & F \\ x_i & \mapsto & f(y_1, ..., y_{i-1}, x_i, y_{i+1}, ..., y_n) \end{array}$$

is linear. In other words, f is multilinear if it is linear with respect to each variable.

*We say that f is **bilinear** if it is multilinear with $n = 2$.*

Theorem 2.2 *Let $(E_1, \|\cdot\|_1)$, $(E_2, \|\cdot\|_2)$, and $(F, \|\cdot\|_F)$ be normed $\mathbb{K}$-vector spaces and let $f : E_1 \times E_2 \to F$ be a bilinear map. The following properties are equivalent:*

(a) f is continuous on $E_1 \times E_2$.

(b) f is continuous at $(0, 0)$.

(c) There exists a constant $C > 0$ such that for all $(x_1, x_2) \in E_1 \times E_2$, we have:

$$\|f(x_1, x_2)\|_F \leq C\|x_1\|_1\|x_2\|_2.$$

Proof.

1. The implication (a) $\Rightarrow$ (b) is obvious.

2. $(b) \Rightarrow (c)$ Suppose $E_1 \times E_2$ is normed by $\|(x_1, x_2)\|_\infty = \max\{\|x_1\|_1, \|x_2\|_2\}$. Assume f is continuous at $(0, 0)$. Then, for $\varepsilon = 1$, there exists $r > 0$ such that for all $(x_1, x_2) \in E_1 \times E_2$, if $\max\{\|x_1\|_1, \|x_2\|_2\} \leq r$, we have:

$$\|f(x_1, x_2)\|_F \leq 1.$$

   Let $(x_1, x_2) \in E_1 \times E_2$ with $x_1 \neq 0$, $x_2 \neq 0$, and define $y_1 = \frac{rx_1}{\|x_1\|_1}$, $y_2 = \frac{rx_2}{\|x_2\|_2}$. Then $\|(y_1, y_2)\|_\infty = r$, hence

$$\|f(y_1, y_2)\|_F = \frac{r^2}{\|x_1\|_1\|x_2\|_2}\|f(x_1, x_2)\|_F \leq 1,$$

   which gives

$$\|f(x_1, x_2)\|_F \leq \frac{1}{r^2}\|x_1\|_1\|x_2\|_2.$$

3. $(c) \Rightarrow (a)$ Suppose there exists $C > 0$ such that $\|f(x_1, x_2)\|_F \leq C\|x_1\|_1\|x_2\|_2$. Fix $(x_0, y_0) \in E_1 \times E_2$ and we will show that f is continuous at (x_0, y_0). Let (x_n, y_n) be a sequence converging to (x_0, y_0) in $E_1 \times E_2$, i.e.,

$$\|(x_n, y_n) - (x_0, y_0)\|_\infty = \max\{\|x_n - x_0\|_1, \|y_n - y_0\|_2\} \to 0,$$

   hence,

$$\|x_n - x_0\|_1 \to 0 \quad \text{and} \quad \|y_n - y_0\|_2 \to 0. \tag{2.5}$$