1.7 Exercises

On norms and norm equivalence

Exercice 1.4 *Let $E = C^1([0,1],\mathbb{R})$ be the vector space of continuously differentiable functions from $[0,1]$ to $\mathbb{R}$. For every $f \in E$, define*

$$\|f\|_1 = \sup_{x\in[0,1]} (|f(x)| + |f'(x)|).$$

1. *Show that $\|\cdot\|_1$ is a norm on E.*
2. *Is it equivalent to the norm $\|f\|_2 = \sup_{x\in[0,1]} |f(x)| + \sup_{x\in[0,1]} |f'(x)|$?*

Exercice 1.5 *Let $E = C([0,1],\mathbb{R})$ be the vector space of continuous functions from $[0,1]$ to $\mathbb{R}$. For every $f \in E$, define*

$$N(f) = \int_0^1 t|f(t)|\,dt \quad \text{and} \quad \|f\|_\infty = \sup_{t\in[0,1]} |f(t)|.$$

1. *Show that N is a norm on E.*
2. *For every $n \in \mathbb{N}$, define:*

$$f_n(t) = \begin{cases} 1 - nt, & \text{if } 0 \le t \le \frac{1}{n}, \\ 0, & \text{if } \frac{1}{n} \le t \le 1. \end{cases}$$

   (a) *Compute $\|f_n\|_\infty$ and $N(f_n)$.*

   (b) *Deduce that the two norms are not equivalent.*

Exercice 1.6 *Let $E = C([0,1],\mathbb{R})$. Define the norms $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$ by*

$$\|f\|_1 = \int_0^1 |f(t)|\,dt, \quad \|f\|_2 = \left(\int_0^1 |f(t)|^2\,dt\right)^{\frac{1}{2}} \quad \text{and} \quad \|f\|_\infty = \sup_{t\in[0,1]} |f(t)|.$$

Show that these three norms are not pairwise equivalent.

Exercice 1.7 (Important Inequalities)

1. *Let $p, q \in (1,+\infty]$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$.*

   (a) *For fixed $s \in (0,+\infty)$, study the variation of the function $\varphi(t) = \dfrac{s^p}{p} + \dfrac{t^q}{q} - st$ on $(0,+\infty)$.*

   (b) *Deduce that for all $s, t \in (0,+\infty)$: $\quad st \le \dfrac{s^p}{p} + \dfrac{t^q}{q}$.*

2. *Let $(x_1, x_2, ..., x_N), (y_1, y_2, ..., y_N) \in \mathbb{R}^N$. Define*

$$s = \frac{|x_k|}{\left(\sum_{k=1}^N |x_k|^p\right)^{1/p}}, \quad t = \frac{|y_k|}{\left(\sum_{k=1}^N |y_k|^q\right)^{1/q}},$$

   establish Hölder's inequality:

$$\left|\sum_{k=1}^N x_k y_k\right| \le \left(\sum_{k=1}^N |x_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^N |y_k|^q\right)^{\frac{1}{q}}.$$

3. (a) *Verify that*

$$\sum_{k=1}^{N}|x_k+y_k|^p \leq \sum_{k=1}^{N}|x_k||x_k+y_k|^{p-1}+\sum_{k=1}^{N}|y_k||x_k+y_k|^{p-1}.$$

   (b) *Using Hölder's inequality, deduce Minkowski's inequality:*

$$\Big(\sum_{k=1}^{N}\Big|x_k+y_k\Big|^p\Big)^{\frac{1}{p}} \leq \Big(\sum_{k=1}^{N}|x_k|^p\Big)^{\frac{1}{p}}+\Big(\sum_{k=1}^{N}|y_k|^p\Big)^{\frac{1}{p}}.$$

4. *For every* $x=(x_1,x_2,...,x_N)\in\mathbb{R}^N$*, define* $\|x\|_p=\Big(\sum_{k=1}^{N}|x_k|^p\Big)^{\frac{1}{p}}$*. Verify that* $\|\cdot\|_p$ *is a norm on* $\mathbb{R}^N$*.*

5. *For every* $p\in[1,+\infty[$*, denote by* ℓ^p *the set of real sequences* $x=(x_n)_{n\geq 0}$ *for which the series* $\sum_{n\geq 0}|x_n|^p$ *converges. Note that* ℓ^p *is a vector space over* $\mathbb{R}$*. For every* $x=(x_n)_{n\geq 0}\in\ell^p$*, define:*

$$N_p(x)=\Big(\sum_{n=0}^{\infty}|x_n|^p\Big)^{\frac{1}{p}}.$$

   Verify that $N_p(\cdot)$ *is a norm on* ℓ^p*.*

On completeness

Exercice 1.8 *Let* $E=C^1([0,1],\mathbb{R})$ *be the space of continuously differentiable real-valued functions on* $[0,1]$*. Define:*

$$\|f\|=\|f\|_\infty+\|f'\|_\infty\ , \qquad \|f\|_1=\|f'\|_\infty+|f(0)|;$$

where $\|f\|_\infty=\sup_{x\in[0,1]}|f(x)|$ *is the sup norm on the space of continuous functions* $F=(C[0,1],\mathbb{R})$*.*

1. *Show that* $\|\cdot\|$ *and* $\|\cdot\|_1$ *are norms on* E*.*

2. *Show that* $\|\cdot\|$ *is equivalent to* $\|\cdot\|_1$*.*

   Hint: *Notice that for all* $x\in[0,1]$*, we have:*

$$f(x)=f(0)+\int_0^x f'(t)\,dt.$$

3. *Knowing that* $(F,\|\cdot\|_\infty)$ *is a Banach space, show that* $(E,\|\cdot\|_1)$ *is a Banach space, and deduce that* $(E,\|\cdot\|)$ *is also a Banach space by following the scheme:*

   (a) *State the Cauchy condition for a sequence* (f_n)*.*

   (b) *Show that the sequence* $(f'_n)_{n\in\mathbb{N}}$ *converges in* $(F,\|\cdot\|_\infty)$ *to some limit* g*, and that the real sequence* $(f_n(0))_{n\in\mathbb{N}}$ *converges in* $(\mathbb{R},|\cdot|)$ *to some limit* l*.*

   (c) *Show that the sequence* (f_n) *converges to* f *in* $(E,\|\cdot\|)$ *where*

$$f(x)=l+\int_0^x g(t)\,dt.$$

(d) *Conclude.*

Exercice 1.9 *Let $E = C([-1,1],\mathbb{R})$ endowed with the L^2 norm:*

$$\|f\|_2 = \left(\int_{-1}^{1} |f(t)|^2\, dt\right)^{1/2}.$$

Consider the sequence (u_n) defined by

$$u_n(x) = \begin{cases} 0, & \text{if } -1 \leq x < \frac{-1}{n}, \\ nx+1, & \text{if } \frac{-1}{n} \leq x < 0, \\ 1, & \text{if } 0 \leq x \leq 1. \end{cases}$$

1. *Show that (u_n) is a Cauchy sequence in E.*
2. *Show that (u_n) does not converge in E.*
3. *What can be deduced?*

Exercice 1.10 *Let $C_0(\mathbb{R},\mathbb{R})$ be the space of functions $f : \mathbb{R} \to \mathbb{R}$ which are continuous and satisfy $\lim\limits_{|x|\to+\infty} f(x) = 0$.*

1. *Show that every function $f \in C_0(\mathbb{R},\mathbb{R})$ is bounded.*
2. *Verify that*
$$\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$$
is a norm on $C_0(\mathbb{R},\mathbb{R})$.
3. *Show that $C_0(\mathbb{R},\mathbb{R})$ is complete under this norm, by following the standard scheme:*
   (a) *State the Cauchy condition for a sequence (f_n).*
   (b) *Show that the sequence (f_n) converges pointwise to a function $f : \mathbb{R} \to \mathbb{R}$.*
   (c) *Show that the sequence (f_n) actually converges uniformly to f.*
   (d) *Deduce that f is continuous and that f tends to 0 at infinity.*
   (e) *Conclude.*

On finite-dimensional spaces

Exercice 1.11 *Let $n \in \mathbb{N}$ and E the vector space of polynomials of degree less than or equal to n. Show that there exists $\lambda > 0$ such that; for every $P \in E$, we have*

$$\int_0^1 |P(x)|\, dx \geq \lambda \sup_{x\in[0,1]} |P(x)|.$$

Exercice 1.12 *Let E be a finite-dimensional normed vector space, K a compact subset of E and $r > 0$. Define*

$$M = \bigcup_{x\in K} \overline{B}(x,r).$$

Show that M is compact.

Exercice 1.13 *Let* $E = C([0, 2\pi], \mathbb{R})$ *endowed with the norm:*

$$\|f\|_2 = \left(\int_0^{2\pi} |f(t)|^2 \, dt \right)^{\frac{1}{2}}.$$

For every $n \in \mathbb{N}$*, define* $f_n(x) = \cos nx$*,* $x \in [0, 2\pi]$*.*

1. *Compute* $\|f_n - f_q\|_2$ *and* $\|f_n\|_2$ *for all* $n, q \in \mathbb{N}$*.*
2. *Show that the sequence* (g_n) *defined by* $g_n = \frac{f_n}{\|f_n\|_2}$ *does not admit any convergent subsequence.*
3. *Deduce that the unit ball* $B(0, 1)$ *is not compact.*
4. *What can be said about the dimension of* E*?*

Exercice 1.14 *Let* F *be a finite-dimensional subspace of a normed vector space* E*.*

1. *Show that for every* $a \in E$*, there exists* $x \in F$ *such that*

$$d(a, F) = \|x - a\|.$$

2. *Suppose* $F \neq E$*. Let* $a \in E \setminus F$ *and let* $x \in F$ *such that* $d(a, F) = \|x - a\|$*. Define*

$$b = \frac{(a - x)}{\|x - a\|}.$$

   Show that $d(b, F) = 1$ *and* $\|b\| = 1$*.*

3. *Suppose* E *is infinite-dimensional. Construct a sequence* (b_n) *in* E *such that, for every* $n \in \mathbb{N}$*,*

$$\|b_n\| = 1 \quad \text{and} \quad d(b_n, Vect\{b_0, b_1, ..., b_{n-1}\}) = 1.$$

4. *Deduce that the closed unit ball of* E *is not compact.*