Chapter 1

Generalities on Normed Vector Spaces

In this chapter, the field of scalars $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$, and the vector spaces will be defined over $\mathbb{K}$.

1.1 Norm on a Vector Space

Definition 1.1 *Let E be a vector space over $\mathbb{K}$. A norm on E is a function:*

$$N : E \rightarrow \mathbb{R}_+ \tag{1.1}$$
$$x \mapsto N(x) \tag{1.2}$$

that satisfies the following properties:

(N1) For all $x \in E$, we have: $N(x) = 0 \Leftrightarrow x = 0$.

(N2) For all $x \in E$ and all $\lambda \in \mathbb{K}$, we have: $N(\lambda x) = |\lambda| N(x)$.

(N3) For all $x, y \in E$, we have: $N(x + y) \leq N(x) + N(y)$.

The norm N will be denoted by the symbol $\|\cdot\|$, and the pair $(E, \|\cdot\|)$ is called a normed space or normed vector space.

Definition 1.2 *Let E be a $\mathbb{K}$-vector space. A* ***normed space*** *is a pair $(E, \|\cdot\|)$ where $\|\cdot\|$ is a norm on E.*

Example 1.1

1. *The absolute value is a norm on $\mathbb{R}$, and the modulus is a norm on $\mathbb{C}$.*
2. *On $\mathbb{R}^n$, the following fundamental norms can be defined:*
   For all $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we have:

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \quad \|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}, \quad \|x\|_\infty = \sup_{1 \leq i \leq n} |x_i|,$$

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \quad \text{for } p \geq 1.$$

   The norm $\|\cdot\|_2$ is the Euclidean norm on $\mathbb{R}^n$.

3. *Let $E = C([a,b],\mathbb{R})$ be the space of real continuous functions on $[a,b]$. For all $f \in E$, the following functions define norms on E:*
   (a) $\|f\|_1 = \int_a^b |f(x)|\,dx$,
   (b) $\|f\|_2 = \left(\int_a^b |f(x)|^2\,dx\right)^{\frac{1}{2}}$,
   (c) $\|f\|_\infty = \max_{a\le x\le b} |f(x)|$,
   (d) $\|f\|_p = \left(\int_a^b |f(x)|^p\,dx\right)^{\frac{1}{p}}$.

Here are some properties of the norm.

Proposition 1.1 1. *Property (N3) above can be replaced by the following:*

(N3') For all $x, y \in E$ and all $t \in [0,1]$, we have:

$$\|tx + (1-t)y\| \le t\|x\| + (1-t)\|y\|.$$

2. *For all $x, y \in E$, we have:*

$$\big|\|x\| - \|y\|\big| \le \|x - y\|. \tag{1.3}$$

3. *Inequality (N3) generalizes to n elements. For all $x_1, x_2, ..., x_n \in E$, we have:*

$$\left\|\sum_{i=1}^n x_i\right\| \le \sum_{i=1}^n \|x_i\|.$$

From inequality (1.3), we get the following statement:

Corollary 1.1 *The norm function $\|\cdot\| : E \to \mathbb{R}^+$ is continuous on E, endowed with the topology induced by $\|\cdot\|$. It is even 1-Lipschitz continuous.*

Remark 1.1 (Distance Associated with a Norm) *Let $(E, \|\cdot\|)$ be a normed space. The following function:*

$$\begin{array}{rcl} f : E \times E & \to & \mathbb{R} \\ (x,y) & \mapsto & d(x,y) = \|x - y\|, \end{array}$$

is called the distance associated with the norm $\|\cdot\|$. It satisfies the following properties:

1. *For all $(x,y) \in E^2$: $d(x,y) = 0 \Leftrightarrow x = y$.*
2. *For all $(x,y) \in E^2$: $d(x,y) = d(y,x)$.*
3. *For all $(x,y,z) \in E^3$: $d(x,z) \le d(x,y) + d(y,z)$.*

The following result gathers some properties of d:

Proposition 1.2 *A normed vector space $(E, \|\cdot\|)$ is a metric space where the distance d is defined by $d(x,y) = \|x - y\|$ for all $x, y \in E$, and we have:*

1. *$d(x+z, y+z) = d(x,y)$ for all $x, y, z \in E$ (i.e., d is translation invariant).*
2. *$d(\lambda x, \lambda y) = |\lambda| d(x,y)$ for all $x \in E$ and all $\lambda \in \mathbb{K}$.*

Proof. Obvious. ■

1.2 Equivalence of Norms

Definition 1.3 *Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space E are said to be* ***equivalent*** *if there exist constants $\alpha, \beta > 0$ such that for all $x \in E$:*

$$\alpha\|x\|_2 \leq \|x\|_1 \leq \beta\|x\|_2.$$

Remark 1.2
- *Two norms on $(E, \|\cdot\|)$ are equivalent if and only if they define equivalent distances, or equivalently, if and only if they define the same topology on E.*
- *For two norms N_1 and N_2 to be equivalent on E, it is necessary and sufficient that:*

$$\exists \alpha, \beta > 0, \forall x \in E: \quad \alpha \leq \frac{N_1(x)}{N_2(x)} \leq \beta \quad \vee \quad \alpha \leq \frac{N_2(x)}{N_1(x)} \leq \beta.$$

Example 1.2
1. *The fundamental norms $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$ defined on $\mathbb{R}^n$ are equivalent.*
2. *Let E be a vector space of finite dimension n and let $\{e_1, e_2, ..., e_n\}$ be a basis of E. Then for every $x \in E$, there exists $(\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{R}^n$ such that:*

$$x = \sum_{i=1}^{n} \lambda_i e_i.$$

The following functions define equivalent norms:

$$\|x\|_1 = \sum_{i=1}^{n} |\lambda_i|, \quad \|x\|_2 = \left(\sum_{i=1}^{n} |\lambda_i|^2\right)^{\frac{1}{2}}, \quad \|x\|_\infty = \sup_{1\leq i\leq n} |\lambda_i|,$$

$$\|x\|_p = \left(\sum_{i=1}^{n} |\lambda_i|^p\right)^{\frac{1}{p}} \quad \text{for } p \geq 1.$$

Remark 1.3 *In general, to prove that two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space E are* ***not*** *equivalent, it suffices to find a sequence $(x_n)_n \subset E$ such that:*

$$\lim_{n\to+\infty} \frac{\|x_n\|_1}{\|x_n\|_2} = 0 \quad or \quad \lim_{n\to+\infty} \frac{\|x_n\|_1}{\|x_n\|_2} = +\infty.$$

For example, the quantity $\|f\|_1 = \int_0^1 |f(t)|\, dt$ is a norm on $C([0,1], \mathbb{R})$ that is not equivalent to the uniform norm $\|f\|_\infty = \max_{0\leq x\leq 1} |f(x)|$. Indeed, consider the sequence of functions $(f_n)_n$ defined by:

$$f_n(x) = \begin{cases} 1 - (n+1)x, & \text{if } 0 \leq x \leq \frac{1}{n+1}; \\ 0, & \text{if } \frac{1}{n+1} \leq x \leq 1. \end{cases}$$

We have $\|f_n\|_\infty = 1$ while $\|f_n\|_1 = \frac{1}{n+1}$, so that:

$$\lim_{n\to+\infty} \frac{\|f_n\|_\infty}{\|f_n\|_1} = \lim_{n\to+\infty} (n+1) = +\infty.$$

1.3 Banach Spaces

Definition 1.4 *A sequence $(x_n)_{n\in\mathbb{N}}$ of elements in a normed space $(E, \|\cdot\|)$ is called a **Cauchy sequence** if:*

$$\forall\varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall p, q \in \mathbb{N},\ p > q \geq n_0 \Rightarrow \|x_p - x_q\| \leq \varepsilon.$$

Definition 1.5 *A sequence $(x_n)_{n\in\mathbb{N}}$ of elements in a normed space $(E, \|\cdot\|)$ is said to **converge** to $x_0 \in E$ if:*

$$\forall\varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow \|x_n - x_0\| \leq \varepsilon.$$

Definition 1.6 *A normed space $(E, \|\cdot\|)$ is said to be **complete** if every Cauchy sequence in E is convergent.*

The advantage of complete spaces is that in such spaces, it is not necessary to know the limit of a sequence to prove its convergence, it suffices to show that it is a Cauchy sequence.

Definition 1.7 *A **Banach space** is a normed space $(E, \|\cdot\|)$ that is complete with respect to the distance induced by the norm.*

Example 1.3 *1. $(\mathbb{R}, |\cdot|)$ and $(C([a,b],\mathbb{R}), \|\cdot\|_\infty)$ are Banach spaces.*

We can verify, for example, that the space $(C([a,b],\mathbb{R}), \|\cdot\|_\infty)$ is complete:

Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. Then:

$$\forall\varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall p, q \in \mathbb{N},\ p > q \geq n_0 \Rightarrow \|f_p - f_q\|_\infty \leq \varepsilon,$$

which implies:

$$\forall p, q \in \mathbb{N},\ p > q \geq n_0,\ \forall x \in [a,b] \Rightarrow |f_p(x) - f_q(x)| \leq \varepsilon. \tag{1.4}$$

Thus, for every $x \in [a,b]$, the sequence $(f_n(x))_{n\in\mathbb{N}}$ of real numbers is a Cauchy sequence. Since $(\mathbb{R}, |\cdot|)$ is complete, this sequence converges to a real number, which we denote by $f(x)$. We define the function:

$$\begin{array}{rcl} f : [a,b] & \to & \mathbb{R} \\ x & \mapsto & f(x) \end{array}$$

Letting $p \to +\infty$ in (1.4), we obtain:

$$\forall q \geq n_0,\ \forall x \in [a,b] \Rightarrow |f(x) - f_q(x)| \leq \varepsilon.$$

Since this inequality holds for all $x \in [a,b]$, it follows that:

$$\|f - f_q\|_\infty \leq \varepsilon.$$

This holds for all $q \geq n_0$, so (f_q) converges to f. Therefore, every Cauchy sequence (f_n) in $C([a,b],\mathbb{R})$ converges, hence $C([a,b],\mathbb{R})$ is complete.

2. *$\mathbb{R}^n$ equipped with $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$ for $p \geq 1$, or with $\|x\|_\infty = \sup_{1\leq i\leq n} |x_i|$, is a Banach space.*

3. *$(C([a,b],\mathbb{R}), \|\cdot\|_1)$ is **not** complete. Indeed, consider the sequence of functions $(f_n)_n$ defined by:*

$$f_n(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq \frac{1}{2}; \\ (x - \frac{1}{2})^{1/n}, & \text{if } \frac{1}{n+1} \leq x \leq 1. \end{cases}$$

The sequence $(f_n)_n$ is a Cauchy sequence but does not converge (with respect to the norm $\|\cdot\|_1$).

Remark 1.4 *Completeness is preserved under change of equivalent norms. For instance, if a sequence $(x_n)_n$ is Cauchy and converges with respect to one norm, then it will also converge with respect to any other equivalent norm.*

1.4 Normed Subspaces and Product Vector Space

Proposition 1.3 (Normed Subspaces) *If F is a vector subspace of a normed space E, the restriction to F of the norm on E is a norm on F, called the induced norm.*

Proof. Obvious, since the norm properties hold for all elements of E, and hence for those of F. The induced norm on F will be denoted as the norm on E. ■

Proposition 1.4 *Let F be a vector subspace of a normed space $(E, \|\cdot\|)$.*

1. *If F is a Banach space with the induced norm, then F is closed in E.*

2. *If E is a Banach space and F is closed in E, then F is a Banach space with the induced norm.*

Proof.

1. Let $x \in \overline{F}$, then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in F converging to x. Consequently, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in F. Since F is complete, $(x_n)_{n\in\mathbb{N}}$ converges to some element $y \in F$. By the uniqueness of the limit, we have $x = y \in F$, hence $\overline{F} = F$. Therefore, F is closed in E.

2. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in F. Then $(x_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence in E, which is complete, so $(x_n)_{n\in\mathbb{N}}$ converges to some element $x \in E$. Since F is closed in E, we have $x \in F$. Therefore, F is complete.

■

Definition 1.8 (Product Vector Space) *Let $(E_1, \|\cdot\|_{E_1})$, $(E_2, \|\cdot\|_{E_2})$ be two normed vector spaces. Then $E = E_1 \times E_2$ is a normed vector space with:*

$$\|x\|_E = \|x_1\|_{E_1} + \|x_2\|_{E_2}, \quad \forall x = (x_1, x_2) \in E.$$

$(E, \|\cdot\|_E)$ is called the product normed space of the given spaces.

This notion can be generalized as follows.

Definition 1.9 (Finite Product of Normed Spaces) *Let $(E_1, \|\cdot\|_{E_1})$, $(E_2, \|\cdot\|_{E_2})$, ..., $(E_n, \|\cdot\|_{E_n})$ be normed spaces. The product set $E = E_1 \times E_2 \times ... \times E_n$ is a normed vector space with:*

$$\|x\|_E = \sum_{i=1}^{n} \|x_i\|_{E_i}, \quad \forall x = (x_1, x_2, ..., x_n) \in E.$$

Proposition 1.5 *Let $(E_1, \|\cdot\|_{E_1})$, $(E_2, \|\cdot\|_{E_2})$, ..., $(E_n, \|\cdot\|_{E_n})$ be normed spaces. For every $x = (x_1, x_2, ..., x_n) \in E = E_1 \times E_2 \times ... \times E_n$, define:*

$$N_1(x) = \|x\|_E = \sum_{i=1}^{n} \|x_i\|_{E_i}, \quad N_2(x) = \left(\sum_{i=1}^{n} \|x_i\|_{E_i}^2\right)^{\frac{1}{2}}, \quad N_\infty(x) = \sup_{1\le i\le n} \|x_i\|_i.$$

Then N_1, N_2, and N_∞ define three equivalent norms on the product vector space E.

We accept the following result:

Proposition 1.6 *Let $(E_1, \|\cdot\|_{E_1})$, $(E_2, \|\cdot\|_{E_2})$, ..., $(E_n, \|\cdot\|_{E_n})$ be normed spaces and $E = E_1 \times E_2 \times ... \times E_n$ the corresponding product normed space. Then E is a Banach space if and only if each E_i is a Banach space for $i \in \{1, 2, ..., n\}$.*

1.5 Normed Spaces of Finite Dimensions

In this paragraph, we are interested in a finite-dimensional $\mathbb{K}$-vector space E of dimension n.

Lemma 1.1 *Let E be a finite-dimensional vector space. Then there exist infinitely many norms on E.*

Proof. Let $\{e_1, e_2, ..., e_n\}$ be a basis of E. Then for every $x \in E$, there exists a unique element $(\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{K}^n$ such that $x = \sum_{i=1}^{n} \lambda_i e_i$. Then the mapping

$$\begin{array}{rcl} \psi : \mathbb{K}^n & \to & E \\ (\lambda_1, \lambda_2, ..., \lambda_n) & \mapsto & \psi(u) = \sum_{i=1}^{n} \lambda_i e_i \end{array}$$

is linear and bijective. Since there are infinitely many norms on $\mathbb{K}^n$, we conclude the result. ■

Theorem 1.1 *Let $(E, \|\cdot\|)$ be a finite-dimensional normed vector space. Then all norms on E are equivalent.*

Proof. Let $\{e_1, e_2, ..., e_n\}$ be a basis of E, where n is the dimension of E. For every $x \in E$, there exists a unique element $(\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{K}^n$ such that

$$x = \sum_{i=1}^{n} \lambda_i e_i$$

We define

$$\|x\|_\infty = \max_{1\le i\le n} |\lambda_i|,$$

then $\|\cdot\|_\infty$ is a norm on E. We will show that the two norms $\|\cdot\|$ and $\|\cdot\|_\infty$ are equivalent. We have:

$$\|x\| = \left\|\sum_{i=1}^{n} \lambda_i e_i\right\| \leq \sum_{i=1}^{n} |\lambda_i|\|e_i\| \leq \max_{1\leq i\leq n} |\lambda_i| \sum_{i=1}^{n} \|e_i\| = \left(\sum_{i=1}^{n} \|e_i\|\right) \|x\|_\infty,$$

Let $c = \sum_{i=1}^{n} \|e_i\| > 0$, then we write

$$\|x\| \leq c\|x\|_\infty. \tag{1.5}$$

On the other hand, since the set

$$\{(\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{K}^n : \max_{1\leq i\leq n} |\lambda_i| = 1\}$$

is compact in $(\mathbb{K}^n, \|\cdot\|_\infty)$, then the sphere

$$S(0,1) = \{x \in E : \|x\|_\infty = 1\}$$

is compact in $(E, \|\cdot\|_\infty)$. The function

$$\begin{array}{rcl} \|\cdot\| : (E, \|\cdot\|_\infty) & \to & \mathbb{R}^+ \\ x & \mapsto & \|x\|; \end{array}$$

is continuous. Indeed, according to Proposition 1.1, for all $x, y \in E$, we have:

$$\Big|\|x\| - \|y\|\Big| \leq \|x - y\| \leq c\|x - y\|_\infty.$$

Consequently, there exists $x_0 \in S(0,1)$ such that

$$\inf_{\|x\|_\infty = 1} \|x\| = \|x_0\| > 0.$$

Let $m = \inf_{\|x\|_\infty = 1} \|x\|$, then $m \leq \|x\|$ if $\|x\|_\infty = 1$.

Let $x \in E$ with $x \neq 0$, then we have:

$$\left\|\frac{x}{\|x\|_\infty}\right\|_\infty = 1,$$

hence

$$m \leq \left\|\frac{x}{\|x\|_\infty}\right\|,$$

but we can see that

$$\left\|\frac{x}{\|x\|_\infty}\right\| = \frac{1}{\|x\|_\infty}\|x\|,$$

which shows that

$$m\|x\|_\infty \leq \|x\|,$$

therefore, there exists $m > 0$ such that for all $x \in E$, we have $m\|x\|_\infty \leq \|x\|$. Hence, the two norms $\|\cdot\|$ and $\|\cdot\|_\infty$ are equivalent. ■

Theorem 1.2 *Let $(E, \|\cdot\|)$ be a normed vector space. The following properties are equivalent:*

(a) *E is finite-dimensional.*

(b) *All norms on E are equivalent.*

(c) *All linear forms are continuous.*

Proof.

- The implication $(a) \Rightarrow (b)$ is exactly Theorem 1.1.

- Let us show the implication $(b) \Rightarrow (c)$. Let $f : E \to \mathbb{R}$ be any linear form. For all $x \in E$, define
$$\|x\|_* = |f(x)| + \|x\|.$$
The application $E \ni x \mapsto \|x\|_*$ defines a norm on E. By hypothesis (b), all norms on E are equivalent, so there exists $\alpha > 0$ such that for all $x \in E$, we have
$$\|x\|_* \leq \alpha \|x\|,$$
hence
$$|f(x)| \leq \alpha \|x\|,$$
and therefore f is continuous.

- To prove the implication $(c) \Rightarrow (a)$, it suffices to prove the contrapositive, i.e., suppose E is infinite-dimensional and show that there exists a non-continuous linear form ($\dim E = +\infty$).

  Let $\{e_n\}_{n\geq 1}$ be an infinite sequence of linearly independent vectors in E, and let V be the vector subspace of E generated by $\{e_n\}_{n\geq 1}$. Let W be an algebraic complement of V in E, and let P be the projection onto V parallel to W. Define a linear form $\varphi : V \to \mathbb{R}$ by:
$$\varphi(e_n) = n\|e_n\|,$$
and therefore:
$$\varphi\left(\sum_{i=1}^{n} \lambda_i e_i\right) = \sum_{i=1}^{n} \lambda_i \|e_i\|.$$
Let $f = \varphi \circ P$, then f is a linear form on E, and we have:
$$f(e_n) = \varphi(e_n) = n\|e_n\|,$$
hence
$$\frac{|f(e_n)|}{\|e_n\|} = n,$$
therefore, $\sup\limits_{x\neq 0} \dfrac{|f(x)|}{\|x\|} = +\infty$.
Consequently, f is not continuous.

■ The following properties result from the equivalence of norms.

Corollary 1.2 *Let E be a $\mathbb{K}$-vector space of finite dimension n. If $\{e_i\}_{i=1,..,n}$ is a basis of E, then the linear map*

$$\begin{array}{rcl} h : \mathbb{K}^n & \to & E \\ (\lambda_1, \lambda_2, ..., \lambda_n) & \mapsto & \sum_{i=1}^{n} \lambda_i e_i \end{array}$$

is a homeomorphism.

Proof. Since the norms on $\mathbb{K}^n$ are equivalent, and the same holds for the space E, we choose the norm $\|\cdot\|_\infty$ on $\mathbb{K}^n$, and for all $x = \sum_{i=1}^{n} \lambda_i e_i \in E$, we choose the norm $\|x\| = \max_{i=1,..,n} |\lambda_i|$. In this case, the application h is an isometry and bijective, hence a homeomorphism from $\mathbb{K}^n$ to E. ■

Corollary 1.3 *Every finite-dimensional subspace of a normed vector space is complete.*

Proof. Let $(E, \|\cdot\|)$ be a normed vector space and let $(F, \|\cdot\|)$ be a finite-dimensional subspace of dimension k ($\|\cdot\|_F = \|\cdot\|$ is the induced norm on F). Fix a basis $\{e_i\}_{i=1,..,k}$ in F. Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(F, \|\cdot\|)$, then it is also Cauchy in $(F, \|\cdot\|_\infty)$, where $\|\cdot\|_\infty$ is the sup norm, since all norms in F are equivalent. For all $n \in \mathbb{N}$, we write:

$$u_n = \sum_{i=1}^{k} u_n^i e_i,$$

Each coordinate sequence $\{u_n^1, u_n^2, ..., u_n^k\}$ is Cauchy in $(\mathbb{K}, |\cdot|)$. Since $\mathbb{K}$ is complete, each coordinate sequence $(u_n^i)_{n\in\mathbb{N}}$ converges to some element u_i in $\mathbb{K}$. To complete the proof, we observe that the sequence $(u_n)_{n\in\mathbb{N}}$ converges to

$$u = \sum_{i=1}^{k} u^i e_i$$

in $(F, \|\cdot\|_\infty)$, and clearly $u \in F$. Let $\varepsilon > 0$. Since each sequence $(u_n^i)_{n\in\mathbb{N}}$ converges to u^i, for all $i = 1, ..., k$, there exists $N_\varepsilon^i \in \mathbb{N}$ such that for all $n \geq N_\varepsilon^i$ we have:

$$|u_n^i - u^i| \leq \varepsilon.$$

Set:

$$N_\varepsilon = \max_{i=1,...,k} N_\varepsilon^i,$$

Then $u_n - u = \sum_{i=1}^{k} (u_n^i - u^i) e_i$. If $n \geq N_\varepsilon$, then:

$$\|u_n - u\|_\infty = \max_{i=1,...,k} |u_n^i - u^i| \leq \varepsilon.$$

which means that $(u_n)_{n\in\mathbb{N}}$ converges to u in the norm $\|\cdot\|_\infty$, and thus in the norm $\|\cdot\|$. This implies the completeness of F. ■

Corollary 1.4 *In a normed space $(E, \|\cdot\|)$, every finite-dimensional vector subspace is closed in E.*

Proof. Let F be a finite-dimensional vector subspace of E. By Corollary 1.3, $(F, \|\cdot\|)$ is a Banach space. Then Proposition 1.4 asserts that F is closed in E. ■

1.6 Compactness in normed spaces

Theorem 1.3 (F. Riesz) *Let $(E, \|\cdot\|)$ be a normed space. Then the following properties are equivalent:*

(a) E is finite-dimensional.

(b) E is a locally compact space.

(c) The closed unit ball $\overline{B}(0,1)$ in $(E, \|\cdot\|)$ is compact.

Proof.

- $(a) \Rightarrow (b)$. Thanks to Corollary 1.2, the space $(E, \|\cdot\|)$ is homeomorphic to $(\mathbb{K}^n, \|\cdot\|_\infty)$, which is locally compact, and therefore $(E, \|\cdot\|)$ is locally compact.

- $(b) \Rightarrow (c)$. If $(E, \|\cdot\|)$ is locally compact, then there exists $r > 0$ such that the closed ball $\overline{B}(0,r)$ is compact. Since the map $E \ni x \mapsto \frac{1}{r}x \in E$ is a homeomorphism, then $\overline{B}(0,1) = \frac{1}{r}\overline{B}(0,r)$ is compact.

- $(c) \Rightarrow (a)$. Suppose the closed unit ball $\overline{B}(0,1)$ is compact. Then there exist finitely many points $a_1, a_2, ..., a_n \in E$ such that:

$$\overline{B}(0,1) = \bigcup_{i=1}^{n} B(a_i, \frac{1}{2}). \tag{1.6}$$

  Let $F = \text{Vect}\{a_1, a_2, ..., a_n\}$ be the finite-dimensional subspace generated by the a_i. By contradiction, we prove that $E = F$; suppose there exists $x \in E$ that is not in F. By Corollary 1.4, F is closed in E, hence $d(x, F) = d > 0$. Therefore, there exists $y \in F$ such that:

$$d \leq d(x,y) = \|x - y\| < 2d.$$

  Let $z = \frac{x-y}{\|x-y\|}$, then $\|z\| = 1$ so $z \in \overline{B}(0,1)$, and from (1.6) there exists some i such that

$$\|z - a_i\| < \frac{1}{2}.$$

  We write:

$$x = y + x - y = y + \|x-y\|z = y + \|x-y\|(a_i + z - a_i) = y + \|x-y\|a_i + \|x-y\|(z - a_i).$$

  Since $y + \|x-y\|a_i \in F$, then we have:

$$d \leq d(x,F) \leq \|x-y\|\|z-a_i\| < \frac{1}{2}\|x-y\|,$$

  hence:

$$2d < \|x-y\|.$$

  This contradicts the choice of y, showing that $E = F$.

■

We can restate the implication $(c) \Rightarrow (a)$ in the following equivalent way:

Corollary 1.5 *In any infinite-dimensional space, the closed unit ball is not compact.*

Corollary 1.6 *Let $(E, \|\cdot\|)$ be a normed space of finite dimension and let F be a subset of E. Then the following properties are equivalent:*

(i) *F is compact.*

(ii) *F is bounded and closed.*