University of M'sila
Faculty of Mathematics and coputer science / Department of Mathematics

Ordinary Differential Equations

(3rd year licence of Mathematics)

Tallab Abdelhamid

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0.1 Introduction

Infinitesimal calculus (differential calculus) has quite naturally led scientists to formulate certain laws of physics in terms of relations between, on the one hand, variables depending on another independent variable (generally time), and, on the other hand, their derivatives. These are what we call ordinary differential equations (abbreviated as ODEs). One of the pioneers in this field was Isaac Newton (1642–1727). Since then, many models in physics have been expressed by means of ODEs (among them, in the seventeenth century, the Euler–Lagrange equations for mechanical systems). Below, we give a non-exhaustive list of models described by ODEs, taken from various areas of science : Biology, Chemistry, Electricity, Electronics, Electrotechnics, Mechanics, and so on.

Definition 0.1. [08]
Let E be a vector space of dimension $m \in \mathbb{N}$. The most general form of an ordinary differential equation (ODE) is

$$F\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^k y}{dx^k}\right) = 0, \qquad k \geq 1, \tag{1}$$

where $y : I \subset \mathbb{R} \to E$, and $F : \mathbb{R} \times E^{k+1} \to E$ is a sufficiently regular function.
The order of the ODE is the integer k corresponding to the highest-order derivative in (1), and its **degree** is the power of that highest-order derivative.

Remark 0.1. 1. If $E = \mathbb{R}$, then (1) is called a scalar ODE.

2. If $E = \mathbb{R}^m$, with $m \geq 2$, then (1) is called a vector ODE ; in this case, one speaks of a differential system.

3. The implicit function theorem guarantees that the ODE (1) can be (locally) put into the explicit normal (resolved) form

$$\frac{d^k y}{dx^k} = G\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^{k-1} y}{dx^{k-1}}\right),$$

provided that $\det(J_F) \neq 0$, where J_F is the Jacobian matrix of F, i.e.,

$$a_{ij} = \frac{\partial F_i}{\partial \left(\dfrac{d^k y_j}{dx^k}\right)}, (i, j) \in \{1, \ldots m\}^2 .$$

4. In the case where the Jacobian matrix is singular everywhere, the ODE (1) in implicit form is called a *Differential-Algebraic Equation* (DAE).

Example 0.1. [08]

The ODE (implicit form)

$$(1-x^2)y' + xy + 1 = 0$$

admits the explicit resolved form

$$y' = \frac{xy+1}{x^2-1},$$

valid on the intervals $]-\infty,-1[$, $]-1,1[$, or $]1,+\infty[$. We obtain solutions on each of these three intervals, but the only C^1 solution (continuous, differentiable, and with continuous derivative) is $y = -x$ on $]-\infty,+\infty[$.

Definition 0.2. Let the scalar ODE be given by

$$F\left(x, u, u', \ldots, u^{(n)}\right) = 0, \ \forall x \in I.$$

1. A function $u : I \subset \mathbb{R} \to \mathbb{R}$ is called a *solution* or an *integral curve* of F if u is n-times differentiable on I and
$$F\left(x, y, y', \ldots, y^{(n)}\right) = 0,$$
2. A *general solution* of order n is a solution containing n arbitrary variables, corresponding to n constants of integration.
3. A *particular solution* is obtained from the general solution by substituting specific values for the constants, often chosen according to initial or boundary conditions.
4. A *singular solution* is a solution that cannot be derived from the general solution.

Example 0.2.

1. Show that $y(x) = c\cos(x)$ is a solution of the equation
$$y'' + y = 0,$$
where c is a constant.

   Solution : We compute $y''(x) = -c\sin(x)$, hence
$$y''(x) + y(x) = -c\sin(x) + c\cos(x) = 0.$$
2. Find the differential equation whose general solution is $y = c\sin(x)$.

Solution : Differentiating, we get $y' = c\cos(x)$. Dividing, we obtain

$$\frac{y'}{y} = \cot(x).$$

3. Form the differential equation whose general solution is $y = C_1(x - C_2)^2$.

Solution : Differentiating twice, we get

$$y' = 2C_1(x - C_2), \qquad y'' = 2C_1.$$

Thus

$$\frac{(y')^2}{y} = 2C_1 = y'',$$

so the desired differential equation is

$$\frac{(y')^2}{y} = 2y''.$$

CHAPITRE 1

Fundamental Results, First-Order Differential Equations

1.1 Existence and Uniqueness of Solutions for Cauchy Problems

In this section, we study the existence and uniqueness of solutions of ODEs by introducing the Cauchy problem and providing the necessary and sufficient conditions for the existence of global solutions.

Cauchy Problem

Definition 1.1. [06]

Let E be a vector space and let the function

$$f : \mathbb{R} \times E \longrightarrow E, \quad (x, y) \longmapsto f(x, y).$$

Solving a Cauchy problem means finding all functions

$$y : \left| \begin{array}{ccc} \mathbb{R} & \longrightarrow & E \\ x & \longmapsto & y(x) \end{array} \right. ,$$

such that

$$\begin{cases} \dfrac{dy}{dx} = y'(x) = f(x, y(x)), \\ y(x_0) = y_0, \quad \text{where } x_0 \in \mathbb{R},\ y_0 \in E. \end{cases} \tag{1.1}$$

$$f : \left| \begin{array}{ccc} \mathbb{R} \times E & \longrightarrow & E \\ (x, y) & \longmapsto & f(x, y) \,. \end{array} \right.$$

1.1.1 Solutions of the Cauchy Problem

Let $E = \mathbb{R}^m$, $I \subset \mathbb{R}$, U an open subset of $\mathbb{R} \times \mathbb{R}^m$, and $f : U \to \mathbb{R}^m$ a continuous mapping.

Definition 1.2. [06] A solution of (1.1) is a differentiable function

$$y : I \subset \mathbb{R} \to \mathbb{R}^m,$$

such that :

1. For all $x \in I$, $(x, y(x)) \in U$.
2. For all $x \in I$, $y'(x) = f(x, y(x))$, with $x_0 \in I$, $(x_0, y_0) \in U$.

Maximal Solutions

Definition 1.3. [10] Let $y : I \to \mathbb{R}^m$ and $\tilde{y} : \tilde{I} \to \mathbb{R}^m$ be two solutions of (1.1).
We say that $\tilde{y}$ is an extension of y if $I \subset \tilde{I}$ and $\forall x \in I$, $\tilde{y}(x) = y(x)$.
A solution $y : I \to \mathbb{R}^m$ is called *maximal* if it admits no extension $\tilde{y} : \tilde{I} \to \mathbb{R}^m$ with $I \subsetneq \tilde{I}$.

Theorem 1.1. *[10] Every solution y can be extended to a maximal solution $\tilde{y}$ (not necessarily unique).*

Global Solutions

Let the open set U be of the form $U = J \times U'$, where J is an interval of $\mathbb{R}$ and U' is an open set of $\mathbb{R}^m$.

Definition 1.4. [10] A *global solution* is a solution defined on the whole interval J.

Remark 1.1. Every global solution is maximal, but the converse is false.

Example 1.1. [10] Consider the equation y'^2 on $U = \mathbb{R} \times \mathbb{R}$. The global solution is $y(x) = 0$. If y does not vanish ($y \neq 0$), then

$$\frac{y'}{y^2} = 1,$$

and by integration we obtain

$$-\frac{1}{y(x)} = x + c,$$

thus

$$y(x) = -\frac{1}{x+c}.$$

This formula defines two solutions, respectively on $(-\infty, -c)$ and $(-c, +\infty)$. These solutions are maximal but not global. In fact, $y(x) = 0$ is the only global solution.

Regularity of Solutions

Definition 1.5. [05] A function is said to be of class C^k if it admits continuous partial derivatives up to order k.

Theorem 1.2. *[05] If $f : U \subset \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ is a function of class C^k, then every solution of* (1.1) *is of class C^{k+1}.*

Definition 1.6. [06]

A solution of (1.1) is a differentiable function $y : I \subset \mathbb{R} \longrightarrow \mathbb{R}^m$, such that :

1. $\forall x \in I,\ (x, y(x)) \in U$.
2. $\forall x \in I,\ y'(x) = f(x, y(x)),\ x_0 \in I,\ (x_0, y_0) \in U$

Definition 1.7. [10] Let $y : I \to \mathbb{R}^m$, $\tilde{y} : \tilde{I} \to \mathbb{R}^m$ be solutions of (1.1).
We say that $\tilde{y}$ is an extension of y if $I \subset \tilde{I}$ and $\forall x \in I, \tilde{y}(x) = y(x)$.
We say that a solution $y : I \to \mathbb{R}^m$ is maximal if y does not admit an extension $\tilde{y} : \tilde{I} \to \mathbb{R}^m$ with $I \subsetneq \tilde{I}$.

Theorem 1.3. *[10] Every solution y can be extended into a maximal solution $\tilde{y}$ (not necessarily unique).*

Existence and Uniqueness Theorems

Lemma 1.1. *[10] A function $y : I \to \mathbb{R}^m$ is a solution of the Cauchy problem* (1.1) *if and only if :*

1. *y is continuous and for all $x \in I$, $(x, y(x)) \in U$.*

2. For all $x \in I$, we have

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt.$$

Definition 1.8. [10] We say that f is *locally Lipschitz in y* if, for every $(x_0, y_0) \in U$, there exists a cylinder

$$C = [x_0 - \alpha,\ x_0 + \alpha] \times B(y_0, r_0) \subset U$$

and a constant $k = k(x_0, y_0) \geq 0$ such that f is k-Lipschitz in y on C, i.e.,

$$\forall (x, y_1), (x, y_2) \in C, \quad \|f(x, y_1) - f(x, y_2)\| \leq k\|y_1 - y_2\|.$$

Remark 1.2. [?] For f to be locally Lipschitz in y, it suffices that $\dfrac{\partial f}{\partial y}$ exists and is continuous.

Theorem 1.4. *[12] If f is locally Lipschitz with respect to y, then the Cauchy problem* (1.1) *admits a unique solution.*

Démonstration. We give the proof in the case $m = 1$, which can easily be generalized. Suppose $f \in C([a, b])$ and satisfies the Lipschitz condition :

$$\forall x \in [a, b],\ \forall y, z \in C([a, b]),\ \exists L > 0 : \quad |f(x, y(x)) - f(x, z(x))| \leq L|y(x) - z(x)|.$$

Then the Cauchy problem admits exactly one solution on $[a, b]$, for every $y_0 \in \mathbb{R}$.

Existence (Picard iterations). Define the sequence of functions (y_k) by

$$\begin{cases} y_0(x) = y_0, \\ y_{k+1}(x) = y_0 + \int_a^x f(t, y_k(t))\, dt. \end{cases}$$

By construction, $y_k \in C^\infty([a, b])$. Moreover,

$$y_{k+1}(x) - y_k(x) = \int_a^x \big(f(t, y_k(t)) - f(t, y_{k-1}(t))\big)\, dt.$$

Using the Lipschitz condition,

$$|y_{k+1}(x) - y_k(x)| \leq L \int_a^x |y_k(t) - y_{k-1}(t)|\, dt \leq L\|y_k - y_{k-1}\|_\infty |x - a|.$$

By induction, we get

$$|y_{k+1}(x) - y_k(x)| \le \frac{L^k|x-a|^k}{k!}\,\|y_1 - y_0\|_\infty \le \frac{L^k|b-a|^k}{k!}\,\|y_1 - y_0\|_\infty.$$

Thus, for $p \ge 1$,

$$|y_{k+p}(x) - y_k(x)| \le \frac{L^k|b-a|^k}{k!}\,\|y_1 - y_0\|_\infty \sum_{i=0}^{p-1} \frac{(L|b-a|)^i}{i!}.$$

Since

$$\sum_{i=0}^{p-1} \frac{(L|b-a|)^i}{i!} \le e^{L|b-a|},$$

we conclude that (y_k) is a Cauchy sequence in $C([a,b])$, hence converges uniformly to some $y \in C([a,b])$ satisfying

$$y(x) = y_0 + \int_a^x f(t, y(t))\,dt.$$

Uniqueness. If y and z are two solutions, then

$$y(x) - z(x) = \int_a^x \big(f(t,y(t)) - f(t,z(t))\big)\,dt.$$

By Lipschitz continuity,

$$|y(x) - z(x)| \le L\int_a^x |y(t) - z(t)|\,dt \le L\|y - z\|_\infty |x-a|.$$

Iterating, we obtain

$$|y(x) - z(x)| \le \frac{L^n|x-a|^n}{n!}\|y - z\|_\infty.$$

Letting $n \to \infty$, we deduce $y = z$. □

Example 1.2. [05] Consider the Cauchy problem

$$\begin{cases} y' = f(x,y), \\ y(x_0) = y_0, \end{cases} \qquad f : \mathbb{R} \to \mathbb{R},\ (x,y) \mapsto y^{1/3}. \tag{1.2}$$

The general solutions of (1.2) are

$$y(x) = \varepsilon\left(\tfrac{2}{3}(x-c)\right)^{3/2}, \quad \varepsilon = \pm 1,$$

together with the identically zero solution.

Case 1 : $y_0 \neq 0$. Here, f is C^1 near (x_0, y_0), so the Cauchy theorem guarantees existence and uniqueness of a solution in a neighborhood of (x_0, y_0). The value of ε is fixed by the sign of y_0. However, if we search for a solution defined on $\mathbb{R}$, we obtain infinitely many solutions by piecing together curves at $y_0 = 0$.

Case 2 : $y_0 = 0$. Here, f is not differentiable near $y_0 = 0$, so the Cauchy theorem does not apply. One can find three solutions :

$$y_1(x) = 0, \quad y_2(x) = \left(\tfrac{2}{3}(x - x_0)\right)^{3/2}, \quad y_3(x) = -\left(\tfrac{2}{3}(x - x_0)\right)^{3/2}.$$

Sufficient Condition for Global Solutions

Theorem 1.5. *[10] Let $f : U \to \mathbb{R}^m$ be continuous on an open product set $U = J \times \mathbb{R}^m$, where $J \subset \mathbb{R}$ is an open interval. Suppose there exists a continuous function $k : J \to \mathbb{R}$ such that, for each fixed $x \in J$, the mapping $y \mapsto f(x, y)$ is Lipschitz with constant $k(x)$ on $\mathbb{R}$. Then every maximal solution of $y' = f(x, y)$ is global (i.e., defined on the whole interval J).*

Example 1.3. [10] Consider

$$y' = x\sqrt{x^2 + y^2} = f(x, y). \tag{1.3}$$

We show that every maximal solution of (1.3) is global.

Indeed, f is defined on $\mathbb{R}^2$. For all $x \in \mathbb{R}$ and $y_1, y_2 \in \mathbb{R}$,

$$|f(x, y_1) - f(x, y_2)| = |x| \cdot \left|\sqrt{x^2 + y_1^2} - \sqrt{x^2 + y_2^2}\right|.$$

Using the inequality

$$|y_1 + y_2| \leq \sqrt{x^2 + y_1^2} + \sqrt{x^2 + y_2^2},$$

we obtain

$$|f(x, y_1) - f(x, y_2)| \leq |x||y_1 - y_2|.$$

Thus, f is Lipschitz in y with constant $Q(x) = |x|$, which is continuous on $\mathbb{R}$. By Theorem 1.5, every maximal solution of (1.3) is global.

Exact Explicit Resolution of Certain ODEs

In this chapter, we provide a qualitative study of ordinary differential equations (ODEs) by first presenting graphical resolution methods, using isoclines and the vector field tangent to the solution curves. Then, we introduce explicitly solvable ODEs, such as separable-variable equations, homogeneous equations, exact equations (with integrating factors), Bernoulli's equation, Lagrangeâ's equation, Riccatiâ's equation, Clairautâ's equation, and so on.

Finally, we present methods using Taylor series and Laplace transforms.

Geometric Interpretation of a Differential Equation

Definition 1.9. [06] Consider a family of curves depending on a parameter c, with equation

$$F(x, y, c) = 0. \tag{1.4}$$

For each value of c there corresponds one curve of the family. The function F admits partial derivatives with respect to x and y. Differentiating equation 1.4 with respect to x, knowing that y is also a function of x, we obtain

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = 0. \tag{1.5}$$

Eliminating the parameter c between equations 1.4 and 1.5 leads to a relation

$$R(x, y, y') = 0.$$

This is a first-order differential equation, which is precisely the differential equation of the family of curves given by 1.4. Given a family of curves depending on a parameter, all these curves are integral curves of the same differential equation under the above assumptions. Conversely, solving a first-order differential equation provides a family of curves depending on a parameter.

Remark 1.3. [06] In general, the curves (C) defined by the equation involving n parameters

$$F(x, y, \lambda_1, \lambda_2, \ldots, \lambda_n) = 0, \tag{1.6}$$

satisfy a differential equation of order n

$$G(x, y, y', y''^{(n)}) = 0,$$

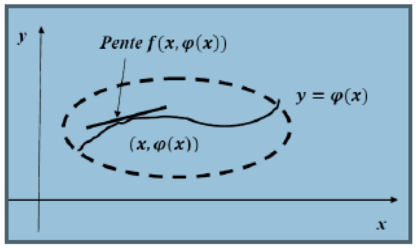

figure 3.1

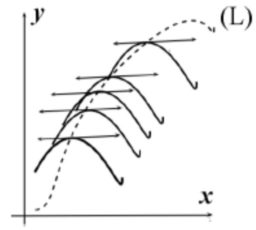

figure 2

obtained by eliminating the n parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$ between equation 1.6 above and the n equations obtained by differentiating it n times with respect to t.

1.1.2 Isoclines

Definition 1.10. [06]

We call *isocline of slope* m for the family of solution curves of the equation

$$y' = f(x, y),$$

the line (L) that is crossed at each of its points by an integral curve with slope equal to m. The isoclines make it possible to determine the shape of the integral curves in the plane.They are defined by the equation

$$f(x, y) = m,$$

where m is a parameter.

The isocline of slope zero is the locus of points with zero slope ; it contains the extremal points of the solution curves (see Figure 3.1 and Figure 3.2).

1.1.3 Graphical Representation of Solutions

Solving a differential equation

$$y' = f(x, y),$$

means determining, either explicitly $y = \varphi(x, c)$ or implicitly $\Phi(x, y, c) = 0$, a family of curves whose tangents are prescribed.

Solving an initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0,$$

amounts to determining the particular curve of this family that passes through a given point.

Thus, one can visualize the problem by plotting the vector field associated with the equation.

Example 1.4. [02] The ODE

$$y' = -\frac{x}{y}$$

The equation can be solved by the method of separation of variables :

1. $\frac{dy}{dx} = -\frac{y}{x}$;
2. $\int ydy = -\int xdx$;
3. $\frac{y^2}{2} = -\frac{x^2}{2} + C$ gives
$x^2 + y^2 = 2C = r^2$.

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The solution $y(x)$ therefore lies on a circle of radius r.

Example 1.5.

Let us Solve the ODE

$$y' = \frac{y^2 - x^2}{2xy}.$$

This equation is homogeneous, since

$$\frac{(\alpha y)^2 - (\alpha x)^2}{2(\alpha x)(\alpha y)} = \frac{y^2 - x^2}{2xy}, \quad \forall \alpha \neq 0.$$

We therefore set $u(x) = y(x)/x$, that is $y(x) = x.u(x)$, and we obtain

The curves $\left(1 + u(x)^2\right).x = C$,

where C is a non-zero real constant

The curves $\left(x - \frac{C}{2}\right)^2 + y(x)^2 = \frac{C^2}{4}$

A family of circles that pass through the origin

and are tangent to the y-axis.)

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Definition 1.11. [02] If we draw at every point (x, y) in the domain of the function $f(x, y)$ a small segment with slope $f(x, y)$, we obtain the direction field of the ODE $y' = f(x, y)$.

Proposition 1.1. *[02] Let f be a solution of the differential equation $y' = f(x, y)$. Then the graph of f is tangent at each of its points (x, y) to the vector $V_{x,y}$.*

Theorem 1.6. *[02] At a point (x, y) we have a solution curve of a differential equation of the form $\frac{dy}{dx} = f(x, y)$. The curve has a tangent $f(x, y)$. We represent the solution graphically by the vector field giving tangent segments to the graph of $f(x, y)$ at the points (x, y). The vector field is given by the following vector function*

$$F(x, y) = \left(\frac{1}{\|(1, \frac{dy}{dx})\|}, \frac{\frac{dy}{dx}}{\|(1, \frac{dy}{dx})\|} \right),$$

which assigns to each point (x, y) a unit vector of length 1 and with the direction of its derivative.

Example 1.6. [02] The direction field of the differential equation

$$\frac{dy}{dx} = y^2 - x^2$$

is the two-dimensional vector

$$F(x, y) = \left(\frac{1}{\sqrt{1 + |y^2 - x^2|^2}}, \frac{y^2 - x^2}{\sqrt{1 + |y^2 - x^2|^2}} \right).$$

Several solution curves are given by the following graph.

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1.2 Separable Equations

This is an equation of the form :

$$y' = G(x, y) \text{ with } G(x, y) = A(x) B(y).$$

For those separable differential equations, we can formally rewrite them in the form ("separation of variables"–each side involve one single variable)

$$\frac{dy}{B(y)} = A(x)\, dx.$$

Integrate both sides with respect to x and y respectively, we have

$$\int \frac{dy}{B(y)} = \int A(x)\,dx.$$

Example 1.7. [06]
Let

$$\left(1+x^2\right)y' = 1 + yavec1 + x^2 > 0,$$

hence

$$\frac{y'}{1+y^2} = \frac{1}{1+x^2},$$

and by integrating, we have

$$\arctan(y) = \arctan(x) + C,$$

then

$$y = \tan\left(\arctan(x) + C\right)$$

1.2.1 Homogeneous Equations

These are equations of the type :

$$y' = f(x,y) = g\left(\frac{y}{x}\right)$$

or in the form

$$P(x,y)\,dx + Q(x,y)\,dy = 0$$

such that P, Q are homogeneous functions.
Knowing that $f(x,y)$ is homogeneous if

$$f(\lambda x, \lambda y) = \lambda^n f(x,y) \text{ for } \lambda \in \mathbb{R}^* \text{ and } n \in \mathbb{N}.$$

Method of solution :
We perform a change of variables

$$y = xu, \text{ then } y' = u + xu' = g(u).$$

We return to an equation with separable variables :

$$\frac{du}{g(u)-u} = \frac{dx}{x}.$$

Example 1.8. [01]

$$xy'^{y/x} = 0$$

This equation is in solved form on $]-\infty, 0[$ or $]0, +\infty[$. Let

$$y = ux \Rightarrow y' = u + xu' \Rightarrow xu'^{u} = 0$$

$$\Rightarrow$$

$$e^{-u}\,du = -\frac{1}{x}\,dx \Rightarrow e^{-u} = \ln(|x|) + C,$$

where C is a constant. We must therefore have $\ln(|x|) + C > 0$ and $u = -\ln(\ln(|x|) + C)$, then $y = -x\ln(\ln(|x|) + C)$.

Example 1.9. Find the general solution of the equation

$$y' = \frac{y}{x} + e^{y/x}.$$

To solve this differential equation, we will first rewrite it to isolate the terms in y and x. The equation is given by :

$$y' = \frac{y}{x} + e^{y/x}.$$

We can treat this equation as a homogeneous first-order differential equation of the form $y' = f(y/x)$. Let $y = ux$ to simplify the equation :

$$\frac{dy}{dx} = u + e^{u}$$

Now, we can solve this homogeneous differential equation using the method of separation of variables or any other appropriate method to find the general solution. Once the solution is found, we can return to the substitution $y = ux$ to obtain the general solution of the original equation. I will now solve the differential equation and get back to you with the general solution.

1.2.2 Equations Reducing to Homogeneous Equations

If

$$y' = f\left(\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}\right).$$

Let $\delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.

1. If $\delta \neq 0$, then in equation **??** we make the substitution

$$\begin{cases} x = u + \alpha \\ y = v + \beta, \end{cases}$$

   where the parameters α and β are determined by the system of equations

$$\begin{cases} a_1\alpha + b_1\beta + c_1 = 0 \\ a_2\alpha + b_2\beta + c_2 = 0, \end{cases}$$

   which reduces to a differential equation with respect to the variables u and v.

2. If $\delta = 0$, then in equation (1) we make the substitution

$$a_1x + b_1y + c_1 = u,$$

   which reduces to a separable differential equation.

Example 1.10. Integrate the equation

$$(2x - y + 4)\,dy + (x - 2y + 5)\,dx = 0.$$

We have

$$\frac{dy}{dx} = -\frac{x - 2y + 5}{2x - y + 4}$$

and

$$\delta = \begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} = -4 + 1 = -3 \neq 0,$$

so by making the change of variables

$$\begin{cases} x = u + \alpha \\ y = v + \beta, \end{cases}$$

with α and β determined by the system of equations

$$\begin{cases} 2\alpha - \beta + 4 = 0 \\ \alpha - 2\beta + 5 = 0, \end{cases}$$

we obtain $\alpha = -1, \beta = 2$, therefore

$$\begin{cases} x = u - 1 \\ y = v + 2, \end{cases}$$

which implies $dx = du, dy = dv$, then

$$\frac{dv}{du} = \frac{2u - v}{u - 2v}$$

is a homogeneous equation with respect to u and v that we know how to solve.

1.2.3 Scalar Linear ODEs

Definition 1.12. [05] Let us consider the following scalar differential equation :

$$F(x, y, y'^{(n)}) = 0.$$

The equation (1.12) is said to be **linear** if it can be written in the form

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_0(x)y(x) - b(x) = 0,$$

where a_i, b_i, and $y : I \subset \mathbb{R} \to K$ are functions of x, with $K = \mathbb{R}$ or $K = \mathbb{C}$.

1.2.4 First-Order Linear ODEs

Theorem 1.7. *[11] Let $y_p(x)$ be a particular solution of the non-homogeneous equation*

$$y'(x) = a(x)y(x) + b(x).$$

Then the general solution of the above differential equation is given by

$$y(x) = y_p(x) + y_h(x),$$

where $y_h(x)$ is the general solution of the associated homogeneous equation

$$y'(x) = a(x)y(x).$$

Démonstration. We have

$$y_p'(x) + y_h'(x) = a(x)y_p(x) + b(x) + a(x)y_h(x) = a(x)(y_p(x) + y_h(x)) + b(x).$$

Conversely, if $y(x) = y_p(x) + y_h(x)$ is a solution of (1.7), then

$$y'(x) - y_p'(x) = a(x)y(x) + b(x) - a(x)y_p(x) - b(x) = a(x)(y(x) - y_p(x)),$$

which is indeed a solution of the homogeneous equation (1.7). □

We have

$$\begin{aligned} & y_p'(x) + y_h'(x) \\ = \;& a(x)\, y_p(x) + b(x) + a(x)\, y_h(x) \\ = \;& a(x)\left(y_p(x) + y_h(x)\right) + b(x), \end{aligned}$$

Conversely, if $y(x) = y_p(x) + y_h(x)$ is a solution of (1.7), then

$$\begin{aligned} & y'(x) - y_p'(x) \\ = \;& a(x)\, y(x) + b(x) - a(x)\, y_p(x) - b(x) \\ = \;& a(x)\left(y(x) - y_p(x)\right), \end{aligned}$$

which is indeed a solution of the homogeneous equation (1.7).

To solve with the initial condition $y(0) = y_0$, the homogeneous equation is separable,

$$\frac{dy}{y} = a(x)dx,$$

and we deduce that the general solution of the homogeneous equation is

$$y_h(x) = y_0 e^{\int_0^x a(t)dt}$$

An elementary way to find the solution of the non-homogeneous equation is the **method of variation of constants**, that is, we look for a solution of the form

$$y\left(x\right)=c\left(x\right)e^{\int_{0}^{x}a(t)dt}.$$

By differentiating, we obtain

$$y'\left(x\right)=c'\left(x\right)e^{\int_{0}^{x}a(t)dt}\;+c\left(x\right)a\left(t\right)e^{\int_{0}^{x}a(t)dt}.$$

Substituting into

$$y'\left(x\right)=a\left(x\right)y+b\left(x\right),$$

the function $c(x)$ satisfies

$$c'\left(x\right)=b\left(x\right)e^{-\int_{0}^{x}a(t)dt},$$

with initial condition $c(0)=y_0$.

Example 1.11. [03] Solve the equation

$$xy'\left(x\right)-y\left(x\right)=x^{2}e^{x},$$

1. on $\left]0,+\infty\right[$.
2. The homogeneous equation has general solution $y(x)=cx$.
3. We seek a particular solution of the form $y(x)=z(x)x$, giving

$$xz\left(x\right)+x^{2}z'\left(x\right)-xz\left(x\right)=x^{2}e^{x}\Leftrightarrow z'\left(x\right)=e^{x}$$

4. Hence $z(x)=e^x+\lambda$, and thus

$$y(x)=\lambda x+xe^{x}.$$

Remark 1.4. Another way to solve the first-order linear ODE (1.7) is to apply the substitution

$$y=uv,\quad \text{where } u,v \text{ are functions of } x.$$

Then

$$y'=u'v+v'u.$$

The ODE (1.7) becomes

$$(u' - a(x)u)v + v'u = b(x).$$

To solve this equation, we require $u' - a(x)u = 0$, from which u is obtained, and then using (1.4), one can find v, and consequently y.

Example 1.12. Consider

$$y' = \frac{y}{x} + x.$$

This is a first-order linear ODE with $a(x) = 1/x$ and $b(x) = x$. Set $y = uv$, so that

$$(u' - \frac{1}{x}u)v + v'u = x.$$

We impose

$$u' - \frac{1}{x}u = 0 \Rightarrow u = x.$$

Substituting into (**??**), we obtain

$$v'x = x \Rightarrow v' = 1 \Rightarrow v = x + c.$$

Hence

$$y = uv = x(x + c).$$

1.2.5 Bernoulli Equations

A **Bernoulli equation** has the form

$$y'(x) = a(x)\,y(x) + b(x)\,y^{\alpha}(x), \;\; \alpha > 0,$$

We can rewrite it as

$$y'(x)\,y^{-\alpha}(x) = a(x)\,y(x) + b(x),$$

Let

$$z(x) = y^{1-\alpha}(x) \Rightarrow z'(x) = (1-\alpha)\,y'(x)\,y^{-\alpha}(x),$$

thus

$$\frac{z'(x)}{1-\alpha} = y'(x)\,y^{1-\alpha}(x),$$

which is a first-order linear ODE that we already know how to solve.

Remark 1.5. 1. If $\alpha = 0$, equation (1.2.5) becomes a linear equation with a non-homogeneous term :

$$y'(x) - a(x)\, y(x) = b(x), \;\; \alpha > 0.$$

2. If $\alpha = 1$, equation (1.2.5) becomes homogeneous :

$$y'(x)\,(b(x) - a(x))\, y(x) = 0.$$

3. One can also use the substitution $y = uv$ as in Remark 1.4.

Example 1.13. Solve

$$y' + 2xy = y^2 e^x.$$

Dividing by y^2, we get

$$y^{-2} y' + 2xy^{-1} = e^x.$$

Let $z = y^{-1} \Rightarrow z'^{-2} y'$. Substituting in (1.13) gives

$$z' - 2xz = -e^x,$$

which is a first-order linear ODE with a non-homogeneous term.

1.2.6 Riccati Equations

A **Riccati equation** has the form

$$y'(x) = a(x)\, y^2(x) + b(x)\, y(x) + c(x).$$

If $y_p(x)$ is a particular solution of (1.2.6), let

$$y(x) = y_p(x) + z(x).$$

Then

$$\begin{aligned} y'(x) &= y_p'(x) + z'(x) \\ &= a(x)\,(y_p(x) + z(x))^2 + (y_p(x) + z(x))\, b(x) + c(x), \end{aligned}$$

which simplifies to

$$\begin{aligned} & y_p'(x) + \ z'(x) \\ = \ & a(x)\, y_p^2(x) + b(x)\, y(x) + c(x) + 2a(x)\, y_p(x)\, z(x) + a(x)\, z^2(x) + b(x)\, z(x), \end{aligned}$$

hence

$$z'(x) = (2a(x)\, y_p(x) + b(x))\, z(x) + \ a(x)\, z^2(x),$$

a Bernoulli's equation with $\alpha = 2$.

Example 1.14. Consider

$$2x^2 y' = (x-1)\left(x^2 - y^2\right) + 2xy \tag{1.7}$$

A particular solution is $y_p = x$. Rewriting as

$$y' = \frac{(1-x)}{2x^2} y^2 + \frac{1}{x} y + (x-1) \tag{1.8}$$

such that$a(x) = \dfrac{(1-x)}{2x^2}$, $b(x) = \dfrac{1}{x}$, and $c(x) = (x-1)$, by setting

$$y = y_p + z(x) = x + z(x), y' = 1 + z'(x),$$

by replacing 1.7, we find

$$z'(x) = \left(\frac{2(1-x)}{2x^2} x + \frac{1}{x}\right) z(x) + \frac{1}{x} z^2(x),$$

it is a Bernoulli's equation.

Proposition 1.2. *Any Riccati differential equation can be transformed into a first-order linear equation.*

Démonstration. Let $\mathbb{R}$be the Riccati equation. Once a particular solution y_p of this equation is known, we perform the following transformations :

Substituting into the Riccati equation and simplifying,

$$y(x) = y_p(x) + \frac{1}{z(x)}, \text{ then } y' = y_p'(x) - \frac{z'}{z^2} \tag{1.9}$$

we obtain, alors

$$y_p'(x) - \frac{z'}{z^2} = a(x)\left[y_p(x) + \frac{1}{z(x)}\right] + b(x)\left[y_p(x) + \frac{1}{z(x)}\right]^2 + c(x)$$

Finally, (after multiplying by z^2) we obtain

$$z' + [2b(x)y_p(x) + a(x)]z = -b(x). \tag{1.10}$$

which is a first-order linear inhomogeneous equation. $\square$

Example 1.15. Solve

$$y' + 3y + y^2 + 2 = 0$$

A particular solution is $y_p = -1$. Using the substitution $y(x) = y_p(x) + \dfrac{1}{z(x)}$, we obtain

$$z' - z = 1.$$

The solution is $z = Ke^x - 1$, $K \in \mathbb{R}$, and thus

$$y = -1 + \frac{1}{Ke^x - 1}, \quad K \in \mathbb{R}.$$

Proposition 1.3. *Consider a Riccati equation of the form*

$$y'^2 + \frac{b}{x}y + \frac{c}{x^2},$$

where a, b, c are constants with $a \neq 0$. If moreover

$$(b+1)^2 - 4ac \geq 0,$$

then this equation admits a particular solution of the form $y_p(x) = \frac{\lambda}{x}$, where

$$\lambda_{1,2} = \frac{-(b+1) \pm \sqrt{(b+1)^2 - 4ac}}{2a}.$$

Remark 1.6. It is not always easy to find a particular solution to a Riccati equation.

Proposition 1.4. *Given a Riccati equation of the form*

$$y' - \frac{1}{2x}y = \frac{A}{x}y^2 + C,$$

where A, B, C are constants, the equation becomes separable under the substitution $y = z\sqrt{x}$.

Démonstration. We observe that $y' - \frac{1}{2x}y = z'\sqrt{x}$. Hence $z'\sqrt{x} = Az^2 + C$, which is a separable equation. □

1.2.7 Exact Differential Equations

Definition 1.13. Consider the equation

$$P(x,y)\,dx + Q(x,y)\,dy = 0.$$

The equation (1.13) is said to be **exact** if its left-hand side represents the total differential of a function $F(x,y)$ in some domain D. The total differential of F is

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy.$$

Corollary 1.1. *1. The general solution of* (1.13) *is* $F(x,y) = c$, *where* F *satisfies* $dF = P\,dx + Q\,dy$.

2. If P, Q *are continuously differentiable in* D, *the equation* (1.13) *is exact if and only if*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Method of solution :

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Then there exists a function F such that

$$P\,dx + Q\,dy = dF = 0.$$

That is,

$$\frac{\partial F}{\partial x} = P, \quad \frac{\partial F}{\partial y} = Q.$$

Integrate the first equation with respect to x :

$$F(x,y) = \int P(x,y)\,dx + c(y).$$

Differentiating with respect to y gives

$$Q(x,y) = \frac{\partial}{\partial y}\left(\int P(x,y)\,dx\right) + c'(y).$$

From (1.2.7), we determine $c(y)$, and substitute it into (1.2.7) to obtain $F(x,y)$.

Remark 1.7. If $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x} = 0$, then the equation is separable. Indeed, P depends only on x and Q only on y, hence $P(x)\,dx = -Q(y)\,dy$.

Example 1.16. [03] Show that the equation

$$y^2 - x^2 + 2xyy' = 0.$$

is exact and solve it. We have

$$(y^2 - x^2)\,dx + 2xy\,dy = 0,$$

hence

$$P(x,y) = y^2 - x^2,\ Q(x,y) = 2xy.$$

Compute :

$$\frac{\partial P}{\partial y} = 2y, \quad \frac{\partial Q}{\partial x} = 2y.$$

Thus the equation is exact.

Integrating $\frac{\partial F}{\partial x} = P$:

$$F(x,y) = xy^2 - \frac{1}{3}x^3 + \lambda.$$

Therefore

$$dF = 0 \iff F(x,y) = k, \quad k \in \mathbb{R},$$

or

$$xy^2 - \frac{1}{3}x^3 = k' \Rightarrow y = \pm\sqrt{\frac{1}{3}x^2 + \frac{k'}{x}}.$$

1.2.8 Integrating Factors

Consider

$$P(x,y)\,dx + Q(x,y)\,dy = 0,$$

and suppose $\dfrac{\partial P}{\partial y} \neq \dfrac{\partial Q}{\partial x}$.

Definition 1.14. [03] A function $\lambda(x, y)$ is called an **integrating factor** of (1.2.8) if there exists a function F such that

$$\frac{\partial F}{\partial x} = \lambda P, \quad \frac{\partial F}{\partial y} = \lambda Q.$$

Then the equation becomes exact if and only if

$$\frac{\partial(\lambda P)}{\partial y} = \frac{\partial(\lambda Q)}{\partial x},$$

that is,

$$\lambda\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = P\frac{\partial \lambda}{\partial y} - Q\frac{\partial \lambda}{\partial x}. \tag{f1}$$

This is a partial differential equation in λ, generally harder to solve than the original one. However, we consider simpler cases : This is a partial differential equation with respect to the unknown function λ. It is generally much more difficult to solve than the original equation. Thus, to solve equation (f1), we consider some simple cases :

1) If λ is a function of x :

Equation (f1) becomes :

$$\frac{dx}{\lambda(x)} = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

and

$$\lambda(x) = \exp\left(\int \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\right).$$

2) If λ is a function of y :

Equation (f1) becomes :

$$\frac{dy}{\lambda(y)} = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

and

$$\lambda(y) = \exp\left(\int \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dy\right).$$

Example 1.17. [03] $y^2 - x^2 - 2xyy' = 0 \Rightarrow$

$$\left(y^2 - x^2\right) dx - 2xydy = 0 \tag{1.11}$$

We verify that $\lambda(x, y) = \dfrac{1}{(x^2 + y^2)^2}$ is an integrating factor of (1.11).

$$\begin{cases} P(x,y) = y^2 - x^2 \\ \\ Q(x,y) = -2xy \end{cases} \Rightarrow$$

$$\begin{cases} \dfrac{\partial P}{\partial y} = 2y \\ \\ \dfrac{\partial Q}{\partial x} = -2y \end{cases}$$

therefore (1.11) is not an exact differential.

Let

$$\begin{cases} \lambda P = \dfrac{y^2 - x^2}{(y^2+x^2)^2} = P_1 \\ \lambda Q = \dfrac{-2xy}{(y^2+x^2)^2} = Q_1 \end{cases}$$

$$\Rightarrow$$

$$\frac{\partial P_1}{\partial y} = \frac{2y(y^2+x^2)^2 - (y^2-x^2)4y(y^2+x^2)}{(y^2+x^2)^4} = \frac{6x^2y - 2y^3}{(y^2+x^2)^3}$$

and

$$\frac{\partial Q_1}{\partial x} = \frac{-2y(y^2+x^2)^2 + 2xy(4y)(y^2+x^2)}{(y^2+x^2)^4} = \frac{6x^2y - 2y^3}{(y^2+x^2)^3}.$$

Hence $\dfrac{\partial P_1}{\partial y} = \dfrac{\partial Q_1}{\partial x}$, therefore

$$\lambda(x,y) = \frac{1}{(x^2+y^2)^2}$$

is an integrating factor of (1.11).

1.3 Lagrange and Clairaut Equations

Lagrange's Equations

These are equations of the form :

$$y = a(y')\,x + b(y') \tag{1.12}$$

where a and b are at least C^1 functions.

Solution method :

Let $y' = p$. This transforms equation (1.12) into :

$$y = a(p)\,x + b(p)\,. \tag{1.13}$$

Differentiating equation (1.13) with respect to x, we obtain :

$$\begin{aligned} p = y' &= a(p) + xp'a'(p) + p'b'(p) \\ &= a(p) + p'\left(xa'(p) + b'(p)\right); \end{aligned}$$

in other words :

$$p - a(p) = p'(xa'(p) + b'(p)).$$

There are two cases :

i) If $p - a(p) = 0$, and its roots are $p_1, p_2, \ldots, p_k$, then :

$$y_i = a(p_i)x + b(p_i) \quad \text{for } 1 \leq i \leq k$$

are solutions of (1.12).

ii) If $p - a(p) \neq 0 \Rightarrow a(p) \neq p$, then :

$$\begin{cases} y = a(p)\,x + b(p) \\ y' = p \end{cases} \Rightarrow \begin{cases} dy = a'(p)\,xdp + a(p)\,dx + b'(p)\,dp \\ dy = pdx \end{cases}$$

$$\Rightarrow$$

$$a'(p)\,xdp + a(p)\,dx + b'(p)\,dp = pdx$$

$$\Rightarrow$$

$$\left(a'(p)\,x + b(p)\right)dp = \left(p - a(p)\right)dx$$

$$\Rightarrow$$

$$\frac{dx}{dp} = \frac{a'(p)}{p - a(p)} x + \frac{b'(p)}{p - a(p)}.$$

Therefore :

$$\frac{dx}{dp} - \frac{a'(p)}{p - a(p)} x = \frac{b'(p)}{p - a(p)}.$$

This equation admits a solution $x = g(p, C)$. It follows that the general solution of (1.12) is of the form :

$$\begin{cases} y &= a(p)\, x + b(p), \\ x &= g(p, C). \end{cases}$$

This solution can be reduced to implicit form $F(x, y, C) = 0$.

Example 1.18. [01] Consider the Lagrange equation :

$$2y - x\left(y' + (y')^3\right) + (y')^2 = 0$$

$$\Rightarrow$$

$$y = \frac{y' + (y')^3}{2} x - \frac{1}{2}(y')^2, \tag{1.14}$$

so :

$$\begin{cases} a(y') = \dfrac{y' + (y')^3}{2} \\ \text{and} \\ b(y') = -\frac{1}{2}(y')^2 \end{cases}.$$

Let $y' = p$, then equation (1.14) transforms to :

$$y = \frac{p + p^3}{2} x - \frac{1}{2} p^2.$$

Differentiating this with respect to x, we find :

$$p - a(p) = p'(x a'(p) + b'(p)).$$

If $a(p) = p$, then

$$\frac{p + p^3}{2} = p \Rightarrow p \in \{0, 1, -1\}$$

$$\Rightarrow$$

$$\begin{cases} y_1 = 0x - 0 = 0 \\ y_2 = \frac{1+1}{2}x - \frac{1}{2} = x - \frac{1}{2} \\ y_3 = \frac{-1-1}{2}x - \frac{1}{2} = -x - \frac{1}{2}. \end{cases}$$

If $a(p) \neq p$:

$$\begin{cases} y = \dfrac{p+p^3}{2}x - \frac{1}{2}p^2 \\ dy = pdx \end{cases}$$

$$\Rightarrow$$

$$\begin{cases} dy = \frac{p \ \ +p^3}{2}dx + \left[\left(\frac{1}{2} + \frac{3}{2}p^2\right)x - p\right]dp \\ dy = pdx \end{cases}$$

$$\Rightarrow$$

$$\frac{dx}{dp} = \frac{1+3p^2}{p-p^3}x - \frac{2p}{p-p^3}$$

a very simple parametric equation (a linear ODE of first order with variable x) to solve.

Example 1.19. Consider the Lagrange equation :

$$y = xy'^2 + y'^2.$$

Let $y' = p$, then the equation transforms to :

$$y = p^2x - (p)^2, \tag{1.15}$$

Differentiating this with respect to x, we find :

$$p - p^2 = p'(2xp + 2p),$$

so :

$$p - p^2 = p'(xa'(p) + b'(p)).$$

If $a(p) = p \Rightarrow p^2 = p \Rightarrow p \in \{0, 1\}$, then :

$$\begin{cases} y_1 = 0x - 0 = 0 \\ y_2 = 1^2x - 1^2 = x - 1. \end{cases}$$

If $a(p) \neq p$:

$$\frac{dx}{dp} = \frac{2p}{p-p^2}x + \frac{2p}{p-p^2} = 2\frac{x+1}{1-p};$$

then :

$$\frac{dx}{x+1} = 2\frac{dp}{1-p};$$

therefore :

$$\ln|x+1| = \ln\left(\frac{1}{(1-p)^2}\right) + c, \quad c \in \mathbb{R},$$

in other words :

$$x+1 = \frac{K}{(1-p)^2}, \quad K \in \mathbb{R},$$

so the general solution of equation (1.15) is the solution of the following parametric equation :

$$\begin{cases} y & = & xa\left(p\right) + b\left(p\right) = \left(x+1\right)p^2 \\ x+1 & = & \dfrac{K}{\left(1-p\right)^2}, \end{cases}$$

$$\Rightarrow p^2 = \frac{y}{x+1},$$

it follows that :

$$x+1 = \frac{K}{\left(1-\sqrt{\dfrac{y}{x+1}}\right)^2};$$

or :

$$x + y - 2\sqrt{y(x+1)} + 1 = K, \quad K \in \mathbb{R}.$$

Clairaut's Equations

Definition 1.15. [01] A Clairaut equation is any differential equation that can be written in the form :

$$y = xy' + f(y')$$

where f is a continuously differentiable function on specified intervals.

Remark 1.8. This is a special case of Lagrange equations if we take $a(y\prime) = y\prime$ and $a(y\prime) = f(y')$ in equation 1.12.

Solution method :

Let $y' = t$, then the equation becomes :

$$y = xt + f(t)$$

Differentiate with respect to x :

$$t = y' = xt' + t + t'f'(t).$$

Hence :

$$xt' + t'f'(t) = 0 = t'(x + f'(t))$$

1) If $t' = 0$, then $t = c \in \mathbb{R}$, hence :

$$y = cx + f(c).$$

2) If $x + f'(t) = 0$, and if f' is invertible, then :

$$t = (f'^{-1}(-x).$$

Example 1.20. Solve $y = xy' + \dfrac{y'}{y'+1}$ on $(-\infty, 0)$.
Let $y' = t$, so the equation becomes :

$$y = xt + \frac{t}{t+1}.$$

Then :

$$y' = xt' + t + \frac{t'(t+1) - tt'}{(t+1)^2} = t$$

$$\Rightarrow$$

$$t'\left(x + \frac{1}{(t+1)^2}\right) = 0.$$

1) If $t' = 0$ then $t = c \in \mathbb{R}$, hence :

$$y = cx + \frac{c}{c+1}$$

is a general solution.

2) If $x + \dfrac{1}{(t+1)^2} = 0$, then :

$$t = \frac{1}{\sqrt{-x}} - 1,$$

then :

$$y = \int \left(\frac{1}{\sqrt{-x}} - 1\right) dx = -x - 2\sqrt{-x}$$

is a singular solution.

Example 1.21. Consider the Clairaut equation :

$$y = xy' - y'^3 \quad \text{and} \quad (f(y') = -y'^3).$$

Let $y' = t$, so the equation becomes :

$$y = xt - t^3.$$

Then :

$$y' = xt' + t - 3t'^2 = t \Rightarrow t'^2) = 0.$$

If $t' = 0$ then $t = c \in \mathbb{R}$, hence :

$$y = cx - c^3.$$

If $x - 3t^2 = 0$, then :

$$t = \frac{1}{\sqrt{3}}\sqrt{x},$$

so :

$$y = \int \frac{1}{\sqrt{3}}\sqrt{x}dx = \frac{2}{3\sqrt{3}}x^{3/2}$$

is a singular solution of the Clairaut equation.

CHAPITRE 2

Methods for Solving Linear Second-Order ODEs

2.1 General Principles

Definition 2.1. [03] Equations of the type

$$y''(x) + a(x)y'(x) + b(x)y(x) = c(x) \tag{2.1}$$

where a, b, c are continuous functions on an interval $I \subset \mathbb{R}$, are called **second-order linear differential equations with variable coefficients**.

If the functions a and b are constants, then equation (2.1) is said to have **constant coefficients**.

If the function $c = 0$, then equation (2.1) is called **homogeneous** (or without a right-hand side). Thus :

$$y''(x) + a(x)y'(x) + b(x)y(x) = 0. \tag{2.2}$$

Definition 2.2. [03] If $a(x), b(x)$ are continuous functions, then equation (2.1) has at least one solution. Furthermore, for any $x_0 \in I$, equation (2.1) has a **unique** solution satisfying the initial conditions :

$$\begin{cases} y(x_0) = y_0 \\ y'(x_0) = y_1. \end{cases}$$

Definition 2.3. [03] Let y_1, y_2 be two solutions of (2.2). We say that y_1, y_2 are **linearly independent** if and only if

$$\lambda_1 y_1 + \lambda_2 y_2 = 0 \quad \Rightarrow \quad \lambda_1 = \lambda_2 = 0.$$

Definition 2.4. [03] We say that y_1, y_2 form a **fundamental system of solutions** for (2.2) if every solution y of (2.2) is a linear combination of y_1 and y_2 ; that is, if y is a solution of (2.2), then

$$y = \lambda_1 y_1 + \lambda_2 y_2.$$

Proposition 2.1. *If y_1 and y_2 are solutions of (2.2), then $y_1 + y_2$ and λy_1 (where λ is a scalar) are also solutions.*

Definition 2.5. Consider the homogeneous equation of type (2.2), and let y_1, y_2 be two solutions of this equation. The **Wronskian** of (y_1, y_2) is defined as the quantity :

$$W_{y_1,y_2}(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}.$$

If $W_{y_1,y_2} \neq 0$, then (y_1, y_2) is called a **fundamental system** for the homogeneous equation (2.2).

Proposition 2.2. *Let $x_0 \in I$. If $W_{y_1,y_2}(x_0) \neq 0$, then $\forall x \in I : W_{y_1,y_2}(x) \neq 0$.*

Corollary 2.1. *Let y_1 and y_2 be solutions of (2.2). Then y_1 and y_2 are linearly independent if and only if $\forall x \in I : W_{y_1,y_2}(x) \neq 0$.*

Theorem 2.1. *If y_1 and y_2 are linearly independent solutions of (2.2), then the general solution of this equation is $y = c_1 y_1 + c_2 y_2$, with $c_1, c_2 \in \mathbb{R}$.*

2.2 Linear Second-Order ODEs with Constant Coefficients

Theorem 2.2. *[11] The general solution y of the non-homogeneous equation*

$$y'' + ay' + by = f(x), \tag{2.3}$$

is the sum of a particular solution y_p of (2.3) and the general solution y_h of the associated homogeneous equation :

$$y = y_p + y_h.$$

Theorem 2.3. *[03] To solve the homogeneous constant coefficient differential equation :*

$$y''(x) + ay'(x) + by(x) = 0,$$

we look for solutions of the form $y_h(x) = e^{rx}$, which leads to the ***characteristic equation*** *:*

$$r^2 + ar + b = 0. \tag{2.4}$$

There are three cases :
1-If equation (2.4) has two distinct real roots r_1 and r_2, the solution is :

$$y_h(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

2-If equation (2.4) has one double root r, the solution is :

$$y_h(x) = e^{rx}(c_1 + c_2 x).$$

3-If equation (2.4) has two complex conjugate roots $\alpha \pm i\beta$, then :

$$y_h(x) = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x).$$

The constants c_1 and c_2 are determined by the initial conditions.

Example 2.1. [03] Consider the equation :

$$y'' + 2y' - 3y = 0.$$

The characteristic equation is :

$$r^2 + 2r - 3 = 0 \Rightarrow r_1 = 1, \quad r_2 = -3.$$

Therefore,

$$y_h = c_1 e^{x} + c_2 e^{-3x}.$$

Remark 2.1. For methods to find a particular solution y_p of (2.3), see the subsection **??**.

2.3 Methods for Solving Linear Second-Order ODEs with a Right-Hand Side

2.3.1 Method of Variation of Parameters

Let y_1, y_2 be two solutions of the homogeneous equation (2.2). The general solution of this equation is :

$$y = c_1 y_1 + c_2 y_2.$$

The principle of this method is to vary the constants c_1 and c_2, treating them as functions of x. That is, we set :

$$y = c_1(x) y_1 + c_2(x) y_2.$$

Differentiating y, we find :

$$y' = c_1'(x) y_1 + c_1(x) y_1' + c_2'(x) y_2 + c_2(x) y_2'.$$

We impose the condition :

$$c_1'(x) y_1 + c_2'(x) y_2 = 0. \tag{2.5}$$

Thus, y' becomes :

$$y' = c_1(x) y_1' + c_2(x) y_2'. \tag{2.6}$$

Differentiating (2.6) gives :

$$y'' = c_1(x) y_1'' + c_2(x) y_2'' + c_1'(x) y_1' + c_2'(x) y_2'.$$

Substituting these expressions into equation (2.3) yields :

$$c_1(x)(y_1'' + a y_1' + b y_1) + c_2(x)(y_2'' + a y_2' + b y_2) + c_1'(x) y_1' + c_2'(x) y_2' = f(x).$$

Since y_1, y_2 are solutions of the homogeneous equation (2.2), the terms in parentheses vanish. This leaves :

$$c_1'(x) y_1' + c_2'(x) y_2' = f(x). \tag{2.7}$$

Equations (2.5) and (2.7) form a linear system for the unknowns $c_1'(x)$ and $c_2'(x)$:

$$\begin{cases} c_1'(x)y_1 + c_2'(x)y_2 = 0 \\ c_1'(x)y_1' + c_2'(x)y_2' = f(x). \end{cases} \tag{2.8}$$

Solving this system allows us to find $c_1'(x)$ and $c_2'(x)$. Integrating them gives $c_1(x)$ and $c_2(x)$, and substituting back into

$$y = c_1(x)y_1 + c_2(x)y_2$$

yields the general solution.

Example 2.2.

Example 2.3. Solve the equation :

$$y'' + y' - 30y = x^2 \tag{2.9}$$

1) Solve the homogeneous equation :

$$y'' + y' - 30y = 0 \tag{2.10}$$

Its characteristic equation is :

$$r^2 + r - 30 = 0,$$

with roots $r_1 = 5$ and $r_2 = -6$. Thus, the general solution of (2.10) is :

$$y_h = c_1 e^{5x} + c_2 e^{-6x}.$$

2) Find y_p using the method of variation of parameters. Solve the system :

$$\begin{cases} c_1'(x)\, e^{5x} + c_2'(x)\, e^{-6x} = 0 \quad (1) \\ 5c_1'(x)\, e^{5x} - 6c_2'(x)\, e^{-6x} = x^2 \quad (2) \end{cases}$$

Multiply equation(1) $\times (-5) + (2)$:

$$-11c_2'^{-6x} = x^2 \quad \Rightarrow \quad c_2'(x) = -\frac{1}{11}x^2 e^{6x}.$$

Substitute into (1) :

$$c_1'(x)\, e^{5x} - \frac{1}{11}x^2 e^{6x} e^{-6x} = 0 \quad \Rightarrow \quad c_1'(x) = \frac{1}{11}x^2 e^{-5x}.$$

Integrating :

$$c_2(x) = \int -\frac{1}{11}x^2 e^{6x} dx = \left(-\frac{1}{66}x^2 + \frac{1}{198}x - \frac{1}{1188}\right) e^{6x},$$

$$c_1(x) = \int \frac{1}{11}x^2 e^{-5x} dx = \left(-\frac{1}{55}x^2 - \frac{2}{275}x - \frac{2}{1375}\right) e^{-5x}.$$

Therefore,

$$y_p = c_1(x)e^{5x} + c_2(x)e^{-6x} = \left(-\frac{1}{30}x^2 - \frac{1}{450}x - \frac{31}{13500}\right).$$

So the general solution is :

$$y_g = y_h + y_p = \left(-\frac{1}{30}x^2 - \frac{1}{450}x - \frac{31}{13500}\right) + K_1 e^{5x} + K_2 e^{-6x}.$$

Remark 2.2.

Remark 2.3. If $W_{y_1,y_2}(x) \neq 0$, we can use Cramer's rule to solve system (2.8) :

$$\begin{aligned} c_1'(x) &= \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{W_{y_1,y_2}(x)} = -\frac{y_2 f(x)}{W_{y_1,y_2}(x)}, \\ c_2'(x) &= \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{W_{y_1,y_2}(x)} = \frac{y_1 f(x)}{W_{y_1,y_2}(x)}. \end{aligned}$$

Thus,

$$c_1(x) = \int -\frac{y_2(x)f(x)}{W_{y_1,y_2}(x)}dx + K_1, \quad c_2(x) = \int \frac{y_1(x)f(x)}{W_{y_1,y_2}(x)}dx + K_2,$$

and the general solution of (2.3) is :

$$y_g = K_1 y_1 + K_2 y_2 - y_1 \int \frac{y_2 f}{W}dx + y_2 \int \frac{y_1 f}{W}dx.$$

2.3.2 Special Cases for Finding y_p (Method of Undetermined Coefficients)

The particular solution y_p of (2.3) can be found by the **method of undetermined coefficients** in the following simple cases :

1. If $f(x) = e^{\alpha x} P_n(x)$, where $P_n(x)$ is a polynomial of degree n.

   i) If α is **not** a root of the characteristic equation (2.4), set

   $$y_p = e^{\alpha x} Q_n(x),$$

   where $Q_n(x)$ is a polynomial of degree n with undetermined coefficients.

   ii) If α **is** a root of the characteristic equation (2.4), set

   $$y_p = x^k e^{\alpha x} Q_n(x),$$

   where k is the multiplicity of the root α ($k = 1$ or $k = 2$).

2. If $f(x) = e^{\alpha x} \left(P_n(x) \cos \beta x + Q_m(x) \sin \beta x \right)$.

   i) If $\alpha \pm i\beta$ is **not** a root of the characteristic equation (2.4), set

   $$y_p = e^{\alpha x} \left(S_N(x) \cos \beta x + T_N(x) \sin \beta x \right),$$

   where $S_N(x)$ and $T_N(x)$ are polynomials of degree $N = \max(n, m)$.

   ii) If $\alpha \pm i\beta$ **is** a root of the characteristic equation (2.4), set

   $$y_p = x^k e^{\alpha x} \left(S_N(x) \cos \beta x + T_N(x) \sin \beta x \right),$$

   where k is the multiplicity of the root $\alpha \pm i\beta$ ($k = 1$ for second-order ODEs).

In all cases, the polynomials can be determined by substituting y_p into the equation and **equating coefficients**.

Example 2.4. 1-Consider the differential equation :

$$y'' + 2y'^2 + 7x - 2.$$

Its characteristic equation is $r^2 + 2r + 3 = 0$. Here, $f(x) = e^{0 \cdot x}(3x^2 + 7x - 2)$, so $\alpha = 0$. Since $\alpha = 0$ is not a root of the characteristic equation, we set :

$$y_p = Ax^2 + Bx + C.$$

Substituting and equating coefficients yields $A = 1, B = 1, C = -2$.
2-Consider the differential equation :

$$y'' - 2y' = x$$

The characteristic equation is $r^2 - 2r + 1 = 0$, which has a double root $r = 1$. The right-hand side is $f(x) = xe^x$, so $\alpha = 1$ and $n = 1$. Since $\alpha = 1$ is a double root, we set :

$$y_p = x^2 e^x (Ax + B).$$

We find $A = \frac{1}{6}, B = 0$. Thus, the general solution is :

$$y_g = (c_1 + c_2 x)e^x + \frac{1}{6}x^3 e^x.$$

3-Consider the differential equation :

$$y'' - 2y' = x.$$

The characteristic equation is $r^2 - 2r + 2 = 0$, with roots $1 \pm i$. The right-hand side is $f(x) = e^x(2\cos x + \sin x)$, so $\alpha \pm i\beta = 1 \pm i$, which is a simple root. Thus, we set :

$$y_p = xe^x(A\cos x + B\sin x).$$

We find $A = -\frac{1}{2}, B = 1$.

2.4 Linear Second-Order ODEs with Variable Coefficients

In practice, we look for a solution of

$$y'' + a(x)y' + b(x)y = f\,(x)\,, \tag{2.11}$$

in the form

$$y = y_h + y_p,$$

where y_h is a solution of the homogeneous equation (H),

$$y'' + a(x)y' + b(x)y = 0,$$

and y_p is a particular solution of 2.11.

Remark 2.4. There is no universal method for finding, in all cases, a solution of 2.11 or of (H). The statement of the problem should guide you.
-One may look for a solution in polynomial form.
-One may also perform a change of variables or a change of unknown function.
-Finally, one may look for a solution expandable in a power series.
-There is only one method to construct the general solution of (1) from a solution of (H) : this is the method of *variation of constants*. However, it may lead to lengthy and difficult computations.

2.4.1 Reduction of Order (First Step)

Let y_1 be a known solution of the homogeneous ODE (2.2). We look for a second, linearly independent solution of the form $y_2 = zy_1$. This implies :

$$y_2' = y_1'z + y_1z', \quad y_2'' = y_1''z + 2y_1'z' + y_1z''.$$

Substituting into (2.2) gives :

$$(y_1'' + ay_1' + by_1)z + z''y_1 + z'(2y_1' + ay_1) = 0.$$

Since y_1 is a solution, the first term vanishes, leaving :

$$z''y_1 + z'(2y_1' + ay_1) = 0. \tag{2.12}$$

To solve (2.12), make the substitution $u = z'$. Then :

$$u'y_1 + u(2y_1' + ay_1) = 0. \tag{2.13}$$

Equation (2.13) is a first-order linear ODE for u, which is straightforward to solve.

2.4.2 Variation of Parameters (Second Step)

If (y_1, y_2) is a fundamental system of solutions for the homogeneous equation, we seek a solution y of the non-homogeneous equation (2.1) in the form :

$$y(x) = \lambda_1(x)y_1(x) + \lambda_2(x)y_2(x).$$

We impose that the derivative satisfies :

$$y'(x) = \lambda_1(x)y_1'(x) + \lambda_2(x)y_2'(x).$$

This expression for y' implies that :

$$\lambda_1'(x)y_1 + \lambda_2'(x)y_2 = 0. \tag{2.14}$$

Substituting y and y' into the differential equation (2.1) yields a second equation. After simplification (using the fact that y_1, y_2 are solutions to the homogeneous equation), we obtain :

$$\lambda_1'(x)y_1' + \lambda_2'(x)y_2' = c(x). \tag{2.15}$$

Equations (2.14) and (2.15) form a linear system for $\lambda_1'(x)$ and $\lambda_2'(x)$:

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} \lambda_1' \\ \lambda_2' \end{pmatrix} = \begin{pmatrix} 0 \\ c(x) \end{pmatrix}.$$

Solving this system gives $\lambda_1'(x)$ and $\lambda_2'(x)$, which can then be integrated to find $\lambda_1(x)$ and $\lambda_2(x)$.

Example 2.5. [03] Solve the non-homogeneous ODE :

$$x^2y'' + xy' - y = 2x,$$

associated with the homogeneous equation :

$$x^2y'' + xy' - y = 0. \tag{2.16}$$

Note that $y_1 = x$ is a solution of (2.16). We look for a second solution of the form $y_2 = zy_1 = xz$. Then :

$$y_2' = z + xz', \quad y_2'' = 2z' + xz''.$$

Substituting into (2.16) :

$$x^2(2z' + xz'') + x(z + xz'^3z''^2z' = 0.$$

Let $u = z'$. Then :

$$x^3u'^2u = 0 \Rightarrow \frac{u'}{u} = -\frac{3}{x} \Rightarrow \ln|u| = -3\ln|x| \Rightarrow u = \frac{1}{x^3} = z'.$$

Integrating :

$$z = \int \frac{1}{x^3}dx = -\frac{1}{2x^2} \Rightarrow y_2 = x\left(-\frac{1}{2x^2}\right) = -\frac{1}{2x}.$$

So the general solution of the homogeneous equation is $y_h = c_1x + c_2\frac{1}{x}$.

Now, for the non-homogeneous equation, we apply variation of parameters :

$$y(x) = \lambda_1(x)x + \lambda_2(x)\frac{1}{x}.$$

The system for $\lambda_1'(x)$ and $\lambda_2'(x)$ is :

$$\begin{cases} \lambda_1'(x)x + \lambda_2'(x)\frac{1}{x} = 0 \\ \lambda_1'(x)\cdot 1 + \lambda_2'(x)\left(-\frac{1}{x^2}\right) = \frac{2x}{x^2} = \frac{2}{x} \end{cases}$$

Solving this system gives :

$$\lambda_1'(x) = \frac{2}{x}, \quad \lambda_2'(x) = -2x.$$

Integrating :

$$\lambda_1(x) = 2\ln|x| + C_1, \quad \lambda_2(x) = -x^2 + C_2.$$

Therefore, the general solution is :

$$y = (2\ln|x| + C_1)x + (-x^2 + C_2)\frac{1}{x} = 2x\ln|x| + C_1x - x + \frac{C_2}{x}.$$

2.5 Power Series Solutions (Frobenius Method)

Let f be a function defined on $I \subset \mathbb{R}$. Then f is expandable in a power series in the neighborhood of $x_0 \in I$ if

$$\exists r > 0, \quad \exists a_n : \mathbb{N} \to \mathbb{R} \text{ a sequence such that} : \forall |x - x_0| < r \Rightarrow f(x) = \sum_{n \geq n_0} a_n (x - x_0)^n.$$

Consider the homogeneous ODE

$$y'' + a(x)y' + b(x)y = 0. \tag{2.17}$$

The goal is to seek solutions of (2.17) in the form of a power series. We assume that a and b are expandable in power series around 0.

Example 2.6. [03] Consider the ODE

$$y'' + xy' + y = 0. \tag{2.18}$$

We set

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

therefore

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substituting into (2.18), we obtain

$$\begin{aligned} y'' + xy' + y &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n a_n x^n + a_1 + \sum_{n=2}^{\infty} a_n x^n + a_0 + a_1 \\ &= a_0 + 2a_1 + \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}x^k + \sum_{k=0}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} a_k x^k = 0 \\ &= 2a_2 + a_0 + \sum_{k=0}^{\infty} \left[(k+2)(k+1)a_{k+2} + k a_k + a_k\right] x^k = 0, \end{aligned}$$

hence

$$\begin{cases} 2a_2 + a_0 = 0, \\ (k+2)(k+1)a_{k+2} + a_k(k+1) = 0, \end{cases}$$

which implies

$$\begin{cases} a_2 = -\frac{1}{2}a_0, \\ a_{k+2} = -\frac{a_k}{k+2}, \end{cases}$$

and thus

$$\begin{cases} a_3 = -\frac{a_1}{3} = \frac{(-1)^2 a_0}{4 \cdot 2}, \\ a_4 = -\frac{a_2}{4} = \frac{(-1)^3 a_1}{5 \cdot 3}. \end{cases}$$

Therefore,

$$a_{2n} = (-1)^n \frac{a_0}{2n(2n-2)(2n-4)\dots 2} \quad \text{and} \quad a_{2n+1} = (-1)^n \frac{a_1}{(2n+1)(2n-1)\dots 1}.$$

Finally,

$$y(x) = c_0 y_0(x) + c_1 y_1(x),$$

where

$$y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n-1)\dots 1}.$$

2.6 Euler's Equation

Definition 2.6. A second-order Euler differential equation is an equation of the form

$$(ax+b)^n y^{(n)} + A_1 (ax+b)^{n-1} y^{(n-1)} + A_2 (ax+b)^{n-2} y^{(n-2)} + ... + A_{n-1} (ax+b) y' + A_n y = f(x)$$

where $a, b, A_1, \dots, A_n$ are real scalars ($a \neq 0$), and g is a function on an interval I.

Solution method :

We set

$$ax + b = e^t,$$

in other words :

$$t = \ln(ax+b).$$

After calculations, we have :

$$\begin{aligned} y' &= ae^{-t}\frac{dy}{dt}, y'' = a^2e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right), \\ y''' &= a^3e^{-3t}\left(\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right), \text{ etc.,} \end{aligned}$$

After this change, the Euler equation transforms into a linear equation with constant coefficients.

Example 2.7. Consider the following differential equation :

$$x^2y'' - xy' + y = 1.$$

We make the change $x = e^t$, we find

$$y' = e^{-t}\frac{dy}{dt}, y'' = e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right).$$

Consequently, the given equation takes the form

$$y'' + y = 1,$$

hence

$$\begin{aligned} y &= c_1 \cos t + c_2 \sin t + 1 \\ &= c_1 \cos(\ln x) + c_2 \sin(\ln x) + 1. \end{aligned}$$

Remark 2.5. For the homogeneous Euler equation

$$x^ny^{(n)} + A_1x^{n-1}y^{(n-1)} + A_2x^{n-2}y^{(n-2)} + \cdots + A_{n-1}xy' + A_ny = 0, \tag{2.19}$$

the solution can be sought in the form

$$y = x^k. \tag{2.20}$$

Substituting into (2.19) the expressions for $y, y'^{(n)}$ given by (2.20), we arrive at a characteristic equation from which we can determine the exponent k.

If k is a real root of the characteristic equation of order m, it corresponds to m linearly independent solutions :

$$y_1 = x^k, \quad y_2 = x^k \ln x, \quad y_3 = x^k \ln^2 x, \quad \ldots, \quad y_m = x^k(\ln x)^{m-1}.$$

If $\alpha \pm i\beta$ is a pair of complex roots of order m, it corresponds to $2m$ linearly independent solutions :

$$\begin{aligned}
y_1 &= x^\alpha \cos(\beta \ln x), y_2 = x^\alpha \sin(\beta \ln x), \\
y_3 &= x^\alpha \ln x \cos(\beta \ln x), y_3 = x^\alpha \ln x \sin(\beta \ln x), \\
&\ldots \\
y_{2m-1} &= x^\alpha (\ln x)^{m-1} \cos(\beta \ln x), y_{2m} = x^\alpha (\ln x)^{m-1} \sin(\beta \ln x)
\end{aligned}$$

Example 2.8. Integrate the equation

$$x^2 y'' - 3xy' + 4y = 0.$$

Let

$$y = x^k, y' = kx^{k-1}, y'' = k(k-1)x^{k-2}. \tag{2.21}$$

Substituting into the given equation and simplifying by x^k, we find the characteristic equation

$$k^2 - 4k + 4 = 0,$$

so

$$k_1 = k_2 = 2, \quad \text{(double root, order } m = 2\text{)},$$

it follows that the general solution is

$$y = c_1 x^2 + c_2 x^2 \ln x.$$

2.7 Cases Where the Order Can Be Reduced

1) If an ODE does not contain y explicitly, for example

$$F(x, y', y'') = 0,$$

by setting $y' = p$, we have an equation whose order is one unit lower :

$$F(x, p, p') = 0.$$

Example 2.9. Find a particular solution of the equation

$$\begin{cases} xy'' + y' + x = 0, \\ y(0) = 0, \\ y'(0) = 0. \end{cases}$$

Setting $y' = p$, we have $y'' = p'$, so

$$xp' + p + x = 0.$$

Integrating this equation as a linear equation with respect to p, we find

$$px = c_1 - \frac{x^2}{2}.$$

We have $y' = p = 0$ for $x = 0$, so $c_1 = 0$, then

$$p = -\frac{x}{2},$$

or

$$\frac{dy}{dx} = -\frac{x}{2}.$$

Integrating once more,

$$y = -\frac{x^2}{4} + c_2.$$

We have $y(0) = 0$, we find $c_2 = 0$. Consequently,

$$y = -\frac{x^2}{4}.$$

2) If an ODE does not contain x explicitly, for example

$$F(y, y', y'') = 0,$$

by setting

$$y' = p, \quad \text{and} \quad y'' = p\frac{dp}{dy},$$

we have an equation whose order is one unit lower :

$$F\left(y, p, p\frac{dp}{dy}\right) = 0.$$

Example 2.10. Find a particular solution of the equation

$$\begin{cases} yy'' + y'^2 = y^4, \\ y(0) = 1, \\ y'(0) = 0. \end{cases}$$

Setting $y' = p$, we have $y'' = p\frac{dp}{dy}$, so

$$yp\frac{dp}{dy} + p^2 = y^4.$$

Integrating this equation as a Bernoulli-type equation with respect to $p(y)$, we find

$$p = \pm y\sqrt{c_1 + y^2}.$$

We have $y' = p = 0$ for $y = 1$, so $c_1 = -1$, then

$$p = \pm y\sqrt{y^2 - 1},$$

or

$$\frac{dy}{dx} = \pm y\sqrt{y^2 - 1}.$$

Integrating, we have

$$\arccos\left(\frac{1}{y}\right) \pm x = c_2.$$

We have $y(0) = 1$, we find $c_2 = 0$. Consequently,

$$\frac{1}{y} = \cos x \quad \Rightarrow \quad y = \frac{1}{\cos x} = \sec x.$$

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