Chapitre 1

Second-order linear partial differential equations

Generalities

Second-order partial differential equations are fundamental tools in applied mathematics. They model physical phenomena such as heat diffusion, wave propagation, and static phenomena. A linear second-order PDE with two independent variables x and y is written in the form:

$$A(x,y)\frac{\partial^2 u}{\partial x^2}+B(x,y)\frac{\partial^2 u}{\partial x\partial y}+C(x,y)\frac{\partial^2 u}{\partial y^2}+D(x,y)\frac{\partial u}{\partial x}+E(x,y)\frac{\partial u}{\partial y}+F(x,y)u=G(x,y), \tag{1.0.1}$$

where $u(x,y)$ is the unknown function, and A,B,C,D,E,F,G are given functions, all of which have at least partial derivatives of order $m\leq 2$ that are continuous on a domain Ω of the x,y plane.

Remarque 1.0.1 *Here* $\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}$ *on the domain* Ω.

- Linear PDEs are said to be homogeneous if $G(x,y)=0$, otherwise they are non-homogeneous.

- They appear in various contexts such as:

1-The one-dimensional wave equation (vibrating string),

$$\frac{\partial^2 u(x,t)}{\partial t^2} - k^2 \frac{\partial^2 u(x,t)}{\partial x^2} = 0$$

2-The heat equation,

$$\frac{\partial u(x,t)}{\partial t} - k^2 \frac{\partial^2 u(x,t)}{\partial x^2} = 0$$

3-The Laplace equation in $\mathbb{R}^2 \frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0$

1.1 Classification of second-order linear EDPs

Définition 1.1.1 ***(Characteristic curves).***

A characteristic for equation (1.0.1) is a curve in Ω satisfying the differential equation:

$$A(\frac{dy}{dx})^2 - B(\frac{dy}{dx}) + C = 0$$

The PDE (1.0.1) is said to be of type:

- Hyperbolic in Ω if at every point in Ω,

$$\Delta = B^2 - 4AC > 0$$

- Parabolic in Ω if at every point in Ω,

$$\Delta = B^2 - 4AC = 0$$

.

- Elliptic in Ω if at every point in Ω,

$$\Delta = B^2 - 4AC < 0.$$

Example

Let us consider the PDE

$$\frac{\partial^2 u}{\partial x^2} + x\frac{\partial^2 u}{\partial y^2} + 2y\frac{\partial u}{\partial x} + u = 0 \tag{1.1.1}$$

$$\Delta = B^2 - 4AC = 0 - 4(1)(x) = -4x$$

then:

The equation (1.1.1) is hyperbolic on the domain $\Omega = \mathbb{R}_*^- \times \mathbb{R}$, (because: $\Delta > 0$).

The equation (1.1.1) is parabolic on the domain $\Omega = \{0\} \times \mathbb{R}$, (because: $\Delta = 0$).

The equation (1.1.1) is elliptic on the domain $\Omega = \mathbb{R}_*^+ \times \mathbb{R}$, (because: $\Delta < 0$).

Lemme 1.1.1 *The classification of hyperbolic, parabolic, or elliptic PDEs is invariant under changes of coordinates.*

Preuve. Let $u = u(\xi, \eta)$ and $\xi = \xi(x, y), \eta = \eta(x, y)$ be two new variables that are functions of x and y with at least their partial derivatives of order $m \leq 2$ continuous on the domain Ω and such that the determinant $J(\zeta, \eta) = \begin{vmatrix} \frac{\partial \xi}{\partial x}, \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x}, \frac{\partial \eta}{\partial y} \end{vmatrix} = \frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} - \frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial y} \neq 0$, on the domain Ω

Using the chain rule, we obtain

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}, \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}(\frac{\partial u}{\partial x}) = \frac{\partial}{\partial x}(\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}) \\
&= \frac{\partial}{\partial \xi}(\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x})\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}(\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x})\frac{\partial \eta}{\partial x} \\
&= (\frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial x})\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + (\frac{\partial^2 u}{\partial \eta \partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x})\frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2} \\
&= \frac{\partial^2 u}{\partial \xi^2}(\frac{\partial \xi}{\partial x})^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}(\frac{\partial \eta}{\partial x})^2 + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2}. \\
\frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial}{\partial x}(\frac{\partial u}{\partial y}) = \frac{\partial}{\partial x}(\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}) \\
&= \frac{\partial}{\partial \xi}(\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y})\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}(\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y})\frac{\partial \eta}{\partial x} \\
&= (\frac{\partial \xi}{\partial x})(\frac{\partial \xi}{\partial y})\frac{\partial^2 u}{\partial \xi^2} + [(\frac{\partial \xi}{\partial x})(\frac{\partial \eta}{\partial y}) + (\frac{\partial \xi}{\partial y})(\frac{\partial \eta}{\partial x})]\frac{\partial^2 u}{\partial \xi \partial \eta} + (\frac{\partial \eta}{\partial x})(\frac{\partial \eta}{\partial y})\frac{\partial^2 u}{\partial \eta^2} + (\frac{\partial^2 \xi}{\partial x \partial y})\frac{\partial u}{\partial \xi} + (\frac{\partial^2 \eta}{\partial x \partial y})\frac{\partial u}{\partial \eta}. \\
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}(\frac{\partial u}{\partial y}) = \frac{\partial}{\partial y}(\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}) \\
&= \frac{\partial}{\partial \xi}(\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y})\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}(\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y})\frac{\partial \eta}{\partial y} \\
&= \frac{\partial^2 u}{\partial \xi^2}(\frac{\partial \xi}{\partial y})^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial y}\frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \eta^2}(\frac{\partial \eta}{\partial y})^2 + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2}.
\end{aligned}$$

Substituting these expressions into equation (1.0.1),the following equation is obtained

$$A'(x,y)\frac{\partial^2 u}{\partial \xi^2} + B'(x,y)\frac{\partial^2 u}{\partial \xi \partial \eta} + C'(x,y)\frac{\partial^2 u}{\partial \eta^2} + D'(x,y)\frac{\partial u}{\partial \xi} + E'(x,y)\frac{\partial u}{\partial \eta} + F'(x,y)u = G'(x,y) \tag{1.1.2}$$

where

$$\begin{aligned}
A' &= A(\frac{\partial \xi}{\partial x})^2 + B\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + C(\frac{\partial \xi}{\partial y})^2, \\
B' &= 2A\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + B(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial y}) + 2C\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y}, \\
C' &= A(\frac{\partial \eta}{\partial x})^2 + B\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + C(\frac{\partial \eta}{\partial y})^2, \\
D' &= A\frac{\partial^2 \xi}{\partial x^2} + B\frac{\partial^2 \xi}{\partial x \partial y} + C\frac{\partial^2 \xi}{\partial y^2} + D\frac{\partial \xi}{\partial x} + E\frac{\partial \xi}{\partial y}, \\
E' &= A\frac{\partial^2 \eta}{\partial x^2} + B\frac{\partial^2 \eta}{\partial x \partial y} + C\frac{\partial^2 \eta}{\partial y^2} + D\frac{\partial \eta}{\partial x} + E\frac{\partial \eta}{\partial y}, \\
F' &= F \text{ et } G' = G
\end{aligned}$$

The equation (1.1.2) is a second-order ODE whose type is determined by the following formula:

$$B'^2 - 4A'C' = (\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} - \frac{\partial\eta}{\partial x}\frac{\partial\xi}{\partial y})(B^2 - 4AC) = J^2(B^2 - 4AC)$$

Since $J \neq 0$, on the domain Ω, the two equations (1.0.1) and (1.1.2) are of the same type. ■

1.2 Characteristic curves and canonical forms (standard)

The canonical form of a second-order linear PDE allows it to be simplified. It is based on the elimination of certain second derivatives. It depends on the type of equation. To do this, we need to introduce the concept of a characteristic curve associated with the equation (1.0.1) .

1.2.1 Characteristic curves

Définition 1.2.1 *The characteristic curves for the equation (1.0.1) are the solutions to the differential equation:*

$$A(\frac{dy}{dx})^2 - B(\frac{dy}{dx}) + C = 0 \tag{1.2.1}$$

given by

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$$

Exercise. Consider the following linear second-order differential equation:

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + 4 = 0$$

1- Identify the type of this equation.

2- Determine the characteristic curves of this equation.

3- Find the corresponding canonical form.

Solution

1- We have $A = 1, B = 0, C = -1$, hence $B^2 - 4AC = 4 > 0$, so the PDE is hyperbolic over all $\mathbb{R}^2$.

2- The characteristic equation is:

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} = \pm 1$$

The associated characteristic curves are:

$$\begin{aligned} \varphi(x, y) &= y + x \\ \psi(x, y) &= y - x \end{aligned}$$

Les courbes caractéristiques associées sont:

$$\begin{aligned} \varphi(x, y) &= y + x \\ \psi(x, y) &= y - x \end{aligned}$$

We set $\xi = y + x, \eta = y - x$ et $u = u(\xi, \eta)$. Using this change of coordinates, by the chain rule we have:

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta} \\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}) \\
&= \frac{\partial}{\partial \xi}(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta})\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta})\frac{\partial \eta}{\partial x} \\
&= \frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \xi \partial \eta} - \frac{\partial^2 u}{\partial \eta \partial \xi} + \frac{\partial^2 u}{\partial \eta^2} \\
&= \frac{\partial^2 u}{\partial \xi^2} - 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}. \\
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}) \\
&= \frac{\partial}{\partial \xi}(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta})\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta})\frac{\partial \eta}{\partial y} \\
&= \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}
\end{aligned}$$

Finally

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + 4 = -4\frac{\partial^2 u}{\partial \xi \partial \eta} + 4 = 0$$

After simplification, we obtain the associated canonical form

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 1.$$

In what follows, we treat each case independently, giving the corresponding form.

1.2.2 Hyperbolic equations

Consider a hyperbolic equation. We know that in this case $\Delta = B^2 - 4AC > 0$ and therefore the equation (1.2.1) has two separate solutions. Thus, there are two real characteristic curves of the equations $\varphi(x, y) = c_1$and $\psi(x, y) = c_2$ or the equation (1.2.1).

Théorème 1.2.1 *Let $\xi = \varphi(x, y), \eta = \psi(x, y),$. The characteristic equations of a hyperbolic PDE are given by:*

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$$

and the canonical form is written as

$$\frac{\partial^2 u}{\partial\xi\partial\eta} = F(\xi, \eta, u, \frac{\partial u}{\partial\xi}, \frac{\partial u}{\partial\eta})$$

Furthermore, if we set $\xi_1 = \xi + \eta$, $\eta_1 = \xi - \eta$ and $u = u(\xi_1, \eta_1)$,we obtain a second canonical form

$$\frac{\partial^2 u}{\partial\xi_1^2} - \frac{\partial^2 u}{\partial\eta_1^2} = G(\xi_1, \eta_1, u, \frac{\partial u}{\partial\xi_1}, \frac{\partial u}{\partial\eta_1})$$

F and G are functions depending on the equations in question.

Example. Let the PDE

$$x^2\frac{\partial^2 u}{\partial x^2} - y^2\frac{\partial^2 u}{\partial y^2} = 0, (x, y) \neq (0, 0)$$

We have $A = x^2, B = 0, C = -y^2$, hense $\Delta = B^2 - 4AC = 4x^2y^2 > 0$. This equation is hyperbolic.

The characteristic equations are:

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{\pm 2xy}{2x^2} = \pm\frac{y}{x}$$

from which

$$y = \frac{c_2}{x}, \quad y = c_1 x$$

so we set

$$\xi = yx, \quad \eta = \frac{y}{x}$$

By the chain rule, we have

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = y\frac{\partial u}{\partial \xi} - \frac{y}{x^2}\frac{\partial u}{\partial \eta}\\
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}(y\frac{\partial u}{\partial \xi} - \frac{y}{x^2}\frac{\partial u}{\partial \eta})\\
&= y\frac{\partial}{\partial x}(\frac{\partial u}{\partial \xi}) + 2\frac{y}{x^3}\frac{\partial u}{\partial \eta} - \frac{y}{x^2}\frac{\partial}{\partial x}(\frac{\partial u}{\partial \eta})\\
&= y[\frac{\partial}{\partial \xi}(\frac{\partial u}{\partial \xi})\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}(\frac{\partial u}{\partial \xi})\frac{\partial \eta}{\partial x}] + 2\frac{y}{x^3}\frac{\partial u}{\partial \eta} - \frac{y}{x^2}[\frac{\partial}{\partial \xi}(\frac{\partial u}{\partial \eta})\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}(\frac{\partial u}{\partial \eta})\frac{\partial \eta}{\partial x}]\\
&= y^2\frac{\partial^2 u}{\partial \xi^2} - \frac{y^2}{x^2}\frac{\partial^2 u}{\partial \xi\partial \eta} + 2\frac{y}{x^3}\frac{\partial u}{\partial \eta} - \frac{y^2}{x^2}\frac{\partial^2 u}{\partial \xi\partial \eta} + \frac{y^2}{x^4}\frac{\partial^2 u}{\partial \eta^2}\\
\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = x\frac{\partial u}{\partial \xi} + \frac{1}{x}\frac{\partial u}{\partial \eta}\\
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}(x\frac{\partial u}{\partial \xi} + \frac{1}{x}\frac{\partial u}{\partial \eta}) = \frac{\partial}{\partial \xi}(x\frac{\partial u}{\partial \xi} + \frac{1}{x}\frac{\partial u}{\partial \eta})\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}(x\frac{\partial u}{\partial \xi} + \frac{1}{x}\frac{\partial u}{\partial \eta})\frac{\partial \eta}{\partial y}\\
&= x^2\frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi\partial \eta} + \frac{1}{x^2}\frac{\partial^2 u}{\partial \eta^2}
\end{aligned}$$

hence the canonical form of this equation is given by:

After simplification, we find

$$-4y^2\frac{\partial^2 u}{\partial \xi\partial \eta} + \frac{2y}{x}\frac{\partial u}{\partial \eta} = 0$$

We have $\xi = yx,\ \ \eta = \dfrac{y}{x}$, hense $y^2 = \xi\eta$,then the canonical form is:

$$\frac{\partial^2 u}{\partial \xi\partial \eta} = \frac{1}{2\xi}\frac{\partial u}{\partial \eta}$$

1.2.3 Parabolic equations

Let us consider a parabolic equation. We know that in this case $\Delta = B^2 - 4AC = 0$ and therefore the equation (1.2.1) has two solutions that coincide. Thus, there is a single real characteristic curve with equation $\varphi(x,y) = c_1$ for the equation (1.2.1) .

Théorème 1.2.2 *Let $\xi = \varphi(x,y), \eta = \psi(x,y)$,where ψ satisfies*

$$J(\xi,\eta) \neq 0$$

the characteristic equation of a parabolic PDE is given by:

$$\frac{dy}{dx} = \frac{B}{2A}$$

and the canonical form is written

$$\frac{\partial^2 u}{\partial \eta^2} = F(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta})$$

Example. Let us consider the EDP

$$\frac{\partial^2 u}{\partial x^2} + 6\frac{\partial^2 u}{\partial x \partial y} + 9\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0 \tag{1.2.2}$$

We have $A = 1, B = 6, C = 9$

Δ=$B^2 - 4AC = (6)^2 - 4 \times 1 \times 9 = 0$. Therefore, equation (1.2.2) is parabolic over all $\mathbb{R}^2$.

The characteristic equation is:

$$\frac{dy}{dx} = \frac{B}{2A} = \frac{6}{2 \times 1} = 3, \text{d'où} \;\; 3x - y = c$$

The associated characteristic curves are therefore

$$\begin{aligned} \xi(x, y) &= 3x - y \\ \eta\,(x, y) &= x \end{aligned}$$

$$J = \begin{vmatrix} 3, -1 \\ 1, \;\; 0 \end{vmatrix} = 1 \neq 0$$

by the chain rule we have

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = 3\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}. \\
\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = -\frac{\partial u}{\partial \xi}. \\
\frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial}{\partial x}(\frac{\partial u}{\partial y}) = \frac{\partial}{\partial \xi}(\frac{\partial u}{\partial y})\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\frac{(\partial u}{\partial y})\frac{\partial \eta}{\partial x} \\
&= 3\frac{\partial u}{\partial \xi}(-\frac{\partial u}{\partial \xi}) + \frac{\partial u}{\partial \eta}(-\frac{\partial u}{\partial \xi}) = -3\frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \xi \partial \eta}. \\
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial \xi}(\frac{\partial u}{\partial x})\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}(\frac{\partial u}{\partial x})\frac{\partial \eta}{\partial x} \\
&= 3\frac{\partial}{\partial \xi}(3\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}) + \frac{\partial}{\partial \eta}(3\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}) \\
&= 9\frac{\partial^2 u}{\partial \xi^2} + 6\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}. \\
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial \xi}(\frac{\partial u}{\partial y})\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}(\frac{\partial u}{\partial y})\frac{\partial \eta}{\partial y} \\
&= -\frac{\partial}{\partial \xi}(-\frac{\partial u}{\partial \xi}) = \frac{\partial^2 u}{\partial \xi^2}
\end{aligned}$$

hence the canonical form of this equation is given by:

$$\frac{\partial^2 u}{\partial \eta^2} = 5\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}$$

1.2.4 Elliptic equations

Let us consider an elliptic equation. We know that in this case $\Delta = B^2 - 4AC < 0$ and therefore the equation (1.2.1) has two complex solutions. Thus, there are two characteristic curves defined from the real and imaginary parts of the solutions.

Théorème 1.2.3 *Let φ be a solution to the equation*

$$\frac{dy}{dx} = \frac{B \pm i\sqrt{4AC - B^2}}{2A}$$

the canonical form is written as

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = F(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta})$$

where

$$\xi = \mathrm{Re}(\varphi), \eta = \mathrm{Im}(\varphi)$$

Example. Consider the PDE

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} + 2\frac{\partial^2 u}{\partial y^2} = 0$$

We have : $A = 1, B = -2, C = 2$

$\Delta = B^2 - 4AC = (-2)^2 - 4 \times 1 \times 2 = -4 < 0$. Therefore, this equation is elliptic.

The characteristic equation is:

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} = -1 \pm i$$

from which

$$y = (-1 + i)x + c_1, \quad y = (-1 - i)x + c_2$$

so we set

$$\xi = y + x, \quad \eta = x$$

Furthermore

$$J = \begin{vmatrix} \frac{\partial \xi}{\partial x}, \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x}, \frac{\partial \eta}{\partial y} \end{vmatrix} = \begin{vmatrix} 1, 1 \\ 1, 0 \end{vmatrix} = -1 \neq 0$$

By the chain rule, we have

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}. \\
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}) = \frac{\partial}{\partial \xi}(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta})\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta})\frac{\partial \eta}{\partial x} \\
&= \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}. \\
\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial u}{\partial \xi}. \\
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}(\frac{\partial u}{\partial \xi}) = \frac{\partial}{\partial \xi}(\frac{\partial u}{\partial \xi})\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}(\frac{\partial u}{\partial \xi})\frac{\partial \eta}{\partial y} = \frac{\partial^2 u}{\partial \xi^2}. \\
\frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial}{\partial x}(\frac{\partial u}{\partial y}) = \frac{\partial}{\partial x}(\frac{\partial u}{\partial \xi}) = \frac{\partial}{\partial \xi}(\frac{\partial u}{\partial \xi})\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}(\frac{\partial u}{\partial \xi})\frac{\partial \eta}{\partial x} \\
&= \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \xi \partial \eta}
\end{aligned}$$

After simplification, we find the canonical form

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = 0.$$