Chapitre 1

Partial differential equations

Introduction

The purpose of partial differential equations (PDEs) is to describe and model physical, natural or economic phenomena that vary according to several independent variables. PDEs can be used to represent heat propagation, fluid motion, substance diffusion, membrane vibrations and many other processes.

For example:

- The heat equation describes how temperature changes in a material over time.
- Laplace's equation models stationary phenomena such as electric potential or pressure distribution in a fluid.
- The wave equation represents the propagation of mechanical waves (sound waves, seismic waves, etc.) or electromagnetic waves.

The main objective of PDEs is therefore to understand and predict the evolution of these phenomena, which is essential in many scientific and engineering disciplines.

Partial differential equations

Définition 1.0.1 *A partial differential equation (PDE) is a mathematical equation that involves partial derivatives of an unknown function u with respect to several independent variables*

$(x_1, x_2, ..., x_n) \in D \subset \mathbb{R}^n$. These derivatives express how the function changes as a function of these variables.

Formally, a PDE is an equation of the form:

$$F(x_1, x_2, ..., x_n, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_2^2}, \frac{\partial^2 u}{\partial x_1 x_2}, ...) = 0 \tag{1.0.1}$$

where:

$u = u(x_1, x_2, \ldots, x_n)$ is the unknown function,

$x_1, x_2, \ldots, x_n$ belong to a certain region Ω of the space of independent variables,

$\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_2^2}, \frac{\partial^2 u}{\partial x_1 x_2}, ...$ are the partial derivatives of u.

Remarque 1.0.1 *Unless otherwise stated, the function u and its partial derivatives involved in the PDE are required to be continuous on Ω.*

Remarque 1.0.2 *Solving an PDE in a domain Ω of $\mathbb{R}^n$ means finding a function u that is sufficiently differentiable in Ω such that (1.0.1) is satisfied for all values of the variables in Ω. If the different solutions of an PDE can be written in the same form, this form is called the general solution of the equation. Since the general solution of an ODE involves arbitrary constants, the general solution of a PDE involves arbitrary functions.*

1.1 Generalities

1.1.1 Linear operator

Let E be a vector space of functions. A linear operator on E is a mapping L defined on E such that:

For all u and v in E and $\alpha, \beta \in \mathbb{R}$

$$L(\alpha u + \beta v) = \alpha L(u) + \beta L(v)$$

$L(u)$ is often denoted Lu.

Example

$$Lu = \frac{\partial u\,(x, y)}{\partial x} + \frac{\partial u\,(x, y)}{\partial y}$$

is linear

Let us take u and v as two functions and $\alpha, \beta \in \mathbb{R}$, then we must verify that

$$L(\alpha u + \beta v) = \alpha Lu + \beta Lv$$

We have

$$\begin{aligned} L(\alpha u + \beta v) &= \frac{\partial (\alpha u + \beta v)}{\partial x} + \frac{\partial (\alpha u + \beta v)}{\partial y} \\ &= \frac{\partial (\alpha u)}{\partial x} + \frac{\partial (\beta v)}{\partial x} + \frac{\partial (\alpha u)}{\partial y} + \frac{\partial (\beta v)}{\partial y} \\ &= \alpha \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) + \beta \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \right) \\ &= \alpha Lu + \beta Lv. \end{aligned}$$

1.1.2 Linear PDE

Définition 1.1.1 *A partial differential equation is said to be linear if it is of the form*

$$Lu = g \tag{1.1.1}$$

where

- L is a linear differential operator.
- g is a function of n independent variables $(x_1, x_2, ..., x_n) \in D \subset \mathbb{R}^n$..
- $u = u(x_1, x_2, ..., x_n)$ is the function sought.
- If $g = 0$, the equation (1.1.1) is said to be homogeneous or without a second member. Otherwise, it is non-homogeneous or with a second member.

1.1.3 Order of a PDE

Définition 1.1.2 *The order of a PDE is the highest order of the partial derivatives involved in the PDE.*

1.1.4 Quasi-linear PDE

Définition 1.1.3 *A PDE is said to be quasi-linear when it is linear with respect to the highest order partial derivatives in each of the variables.*

Examples

1.The equation: $u + y\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial y^2} = 1$, is a linear non-homogeneous PDE of order 2 (for $D = \mathbb{R}^2$), where $u = u(x, y)$.

2. The equation: $\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = \cos(u)$, is a first-order nonlinear PDE.

3. Laplace's equation: $\Delta u = u_{xx} + u_{yy} = 0$, is a second-order, linear, homogeneous PDE

4. The transport equation: $\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = 0$, is a first-order, quasi-linear, homogeneous PDE.

5. The wave equation: $\dfrac{\partial^2 u}{\partial t^2} - c(u)\dfrac{\partial^2 u}{\partial x^2} = 0$, is a second-order, quasi-linear PDE.

Remarque 1.1.1 *For a linear homogeneous PDE $Lu = 0$. If u_1 and u_2 are two solutions to this PDE and $\alpha, \beta \in \mathbb{R}$, then the linear combination $\alpha u_1 + \beta u_2$ is also a solution.*

Derivatives of a function composed of two variables

Théorème 1.1.1 *Let $u = u(\xi, \eta)$, where $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$, then*

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \xi}.\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}.\frac{\partial \eta}{\partial x} \\
\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial \xi}.\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}.\frac{\partial \eta}{\partial y}
\end{aligned}$$

Example

$u(\xi, \eta) = \xi^2 + \xi\eta + \eta^2$, where $\xi(x, y) = 2x + y$, and $\eta(x, y) = 2x - y$

We have

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \xi}.\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}.\frac{\partial \eta}{\partial x} = 6\xi + 6\eta \\
\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial \xi}.\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}.\frac{\partial \eta}{\partial y} = \xi - \eta.
\end{aligned}$$

1.1.5 Total differential

Définition 1.1.4 *Let $u(x,y)$ be a function of two variables with continuous partial derivatives. The total or exact differential of $u(x,y)$ is written as:*

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

Example. Let us consider the function: $u(x,y) = x^2 + xy^3$, then

$$du = (2x + y^3)dx + (3xy^2)dy.$$

1.1.6 First-order PDE

Définition 1.1.5 *An equation containing a function u of several independent variables $x, y, ...$ and first-order partial derivatives of u with respect to these variables,*

i.e. an equation of the form

$$F(x, y, ...,, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, ...) = 0$$

is called a first-order PDE.

In the case of two variables x, y, we have

$$F(x, y, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = 0$$

- $\dfrac{\partial u}{\partial x} = 0 \Leftrightarrow u(x,y) = f(y)$
- $\dfrac{\partial u}{\partial y} = 0 \Leftrightarrow u(x,y) = f(x)$
- $\dfrac{\partial u}{\partial x} = h(x)$

If g is integrable and H is one of its primitives, then

$$\frac{\partial}{\partial x}(u(x,y) - H(x)) = 0$$

from which

$$u(x, y) = H(x) + f(y)$$

where f is an arbitrary function.

Remarque 1.1.2 *The most general form for a first-order linear PDE with two variables is:*

$$A(x, y)\frac{\partial u}{\partial x} + B(x, y)\frac{\partial u}{\partial y} + C(x, y)u = D(x, y)$$

where $A(x, y), B(x, y)$, $C(x, y)D$ and (x, y) are given functions and u is the function being sought.

1.2 Solving first-order PDEs

Définition 1.2.1 *(Parametric curve)*

A parametric curve in a plane is defined by two functions $x(t)$ and $y(t)$, where t varies within a certain interval. The points on the curve are given by: $C(t) = (x(t), y(t))$ for $t \in [a, b]$

$x(t)$ is the abscissa (the coordinate on the x-axis).

$y(t)$ is the ordinate (the coordinate on the y-axis).

The parameter t is used to describe the evolution of the x and y coordinates on the curve.

Example (Circle)

For a circle of radius r centred at the origin, we cannot directly express y as a function of x in a simple way (this would give $y = \pm\sqrt{r^2 - x^2}$. But with parameterisation, we can write the following equations:

$$\begin{aligned} x(t) &= r\cos(t) \\ y(t) &= r\sin(t) \end{aligned}$$

with $t \in [0, 2\pi]$, which represents a complete turn of the circle.

1.2.1 Characteristic method

The characteristic method is based on the idea of finding curves (characteristics) along which the PDE reduces to an ordinary differential equation. The characteristics are defined by a system of ordinary differential equations.

1.2.2 Linear PDE case

A linear PDE of order 1 can be written in the form:

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} + c(x,y)u = d(x,y)$$

where $a(x,y), b(x,y)$,$c(x,y)$ and $d(x,y)$are given functions, and $u = u(x,y)$ is the unknown function.

1-Introduction of characteristic curves:

We are looking for curves in the (x, y) plane along which the PDE reduces to an ordinary differential equation. These curves are called characteristics. The equations of the characteristics are obtained from the PDE by assuming that along these curves, the solution remains constant.

To find the equations of the characteristics, we write:

$$\frac{dx}{ds} = a(x,y), \frac{dy}{ds} = b(x,y)$$

where s is a parameter along the characteristics. This system of differential equations determines the characteristic curves.

2-Solving the differential system:

Once the equations of the characteristics have been determined, this system is solved to obtain the characteristic curves. The solutions to these equations give expressions for x(s) and y(s).

3-Ordinary differential equation along the characteristics:

Substituting the expressions for x(s) and y(s) into the initial ODE gives an ordinary differential equation for u along the characteristics:

$$\frac{du}{ds} = d(x(s), y(s)) - c(x(s), y(s))u(s).$$

We solve this equation for u, which gives the solution of the PDE along the characteristic curves.

4-Obtaining the general solution:

The general solution of the PDE is then obtained by combining the solutions found for $x(s), y(s)$, and $u(s)$. If initial or boundary conditions are given, they are used to determine the integration constants.

Example. Consider the following EDP:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

we have: $a(x, y) = 1, b(x, y) = 1, d(x, y) - c(x, y)(u(x, y)) = 0$. Let us apply the method of characteristics.

Equations of characteristics:

$$\begin{cases} \dfrac{dx}{ds} = 1, \\ \dfrac{dy}{ds} = 1 \\ \dfrac{du}{ds} = 0 \end{cases}$$

Integrating these equations, we obtain: $x(s) = s + c_1, y(s) = s + c_2$, and $x - y = c$

The equation: $\dfrac{du}{ds} = 0$ implies that u is constant along the characteristics. Therefore:

$$u = f(c)$$

where f is an arbitrary function to be determined from the initial conditions. The general solution is therefore:

$$u(x,y) = f(x-y)$$

Exercise

Using the characteristic method, solve the following PDEs:

1- $\quad x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = u$

2-$\dfrac{\partial u}{\partial x} + 2\dfrac{\partial u}{\partial y} = 0$, with initial condition: $u(x,0) = x^2$

Solution

1- Characteristic equations:

$$\begin{cases} \dfrac{dx}{ds} = x, \\ \dfrac{dy}{ds} = y \\ \dfrac{du}{ds} = u \end{cases}$$

Integrating these equations, we obtain:

$$x(s) = c_1 e^s, y(s) = c_2 e^s, u(s) = c_3 e^s$$

Since the variables x and y evolve similarly as a function of s, we can introduce a new variable $c = (\dfrac{x}{y})$, which is a constant on the characteristic curves.

This means that the solution u depends on a certain function f of c, i.e.:

$$u(x,y) = f(\frac{x}{y})$$

The general solution of the PDE is therefore:

$$u(x,y) = f(\frac{x}{y})$$

where f is an arbitrary function determined by initial or boundary conditions

2- Characteristic equations:

$$\begin{cases} \dfrac{dx}{ds} = 1, \\ \dfrac{dy}{ds} = 2 \\ \dfrac{du}{ds} = 0 \end{cases}$$

Integrating these equations, we obtain: $x(s) = s + c_1, y(s) = 2s + c_2, u(s) = u_0$
Eliminating s from the equations for x and y, we obtain: $x - (\frac{y}{2}) = c_1 - \frac{c_2}{2} = c$
Thus, the solution u is constant along the characteristics and depends only on c, i.e.:

$$u(x, y) = f(x - \frac{y}{2})$$

The initial condition is $u(x, 0) = x^2$. Applying this, we have:

$$u(x, 0) = f(x) = x^2$$

Therefore,

$$f(x - \frac{y}{2}) = (x - \frac{y}{2})^2$$

The solution to the PDE is therefore:

$$u(x, y) = (x - \frac{y}{2})^2$$

1.2.3 Quasi-linear PDE case

Définition 1.2.2 . *A first-order quasi-linear equation is an equation of the form:*

$$a(x, y, u)\frac{\partial u}{\partial x} + b(x, y, u)\frac{\partial u}{\partial y} = c(x, y, u)$$

where $u = u(x, y)$ is the unknown function, and a, b, and c are given functions.

1.2.4 Characteristic curves:

We are looking for curves $(x(s), y(s))$ in the plane, parameterised by a parameter s, along which $u(x, y)$ satisfies a system of ODEs. The system associated with the PDE is:

$$\begin{cases} \frac{dx}{ds} = a(x, y, u), \\ \frac{dy}{ds} = b(x, y, u), \\ \frac{du}{ds} = c(x, y, u) \end{cases}$$

This system must be solved to obtain the characteristic curves $x(s), y(s)$, and $u(s)$.

The PDE is usually accompanied by an initial condition in the form $u(x_0, y_0) = u_0$. This condition is used to determine the integration constants when solving ODEs.

Geometric interpretation:

The characteristics are curves in space(x, y, u) along which the solution propagates. By following these curves, the solution to the PDE can be constructed.

1.2.5 Non-linear case

Définition 1.2.3 *.A first-order non-linear PDE is an equation of the form:*

$$G(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = 0 \tag{1.2.1}$$

we set $p = \frac{\partial u}{\partial x}$,$q = \frac{\partial u}{\partial y}$,the equation (1.2.1) is written as

$$G(x, y, u, p, q) = 0$$

The solution to the PDE(1.2.1) is based on the solution to the following characteristic system:

$$\frac{dx}{G_p} = \frac{dy}{G_q} = \frac{du}{pG_q + qG_q} = \frac{-dp}{G_x + pG_u} = \frac{-dq}{G_y + qG_u}$$

where $G_k = \frac{dG}{dk}, k = x, y, p, q, u.$

Example: Solve the following PDE:

$$\frac{\partial u}{\partial x} . \frac{\partial u}{\partial y} = 1, \;\; u(x, 0) = x$$

Let $p = \frac{\partial u}{\partial x}, \;\; q = \frac{\partial u}{\partial y}$ and note that $G(x, y, u, p, q) = pq - 1$

Then $G_x = G_y = G_u = 0, G_p = q, G_q = p,$. The associated characteristic system is then written as:

$$\begin{cases} \frac{dx}{q} = \frac{dy}{p} = \frac{du}{2pq} \\ dp = dq = 0, \end{cases}$$

From $dp = dq = 0$, we obtain $p = c_1, q = c_2$
Since$pq = 1$, from the equation $\dfrac{dx}{q} = \dfrac{dy}{p}$ we obtain $\dfrac{pdx}{pq} = \dfrac{qdy}{qp} = \dfrac{pdx + qdy}{2pq} = \dfrac{du}{2pq}$
from which $du = pdx + qdy$and therefore

$$u(x, y) = c_1 x + \frac{1}{c_1} y + c_3$$

we have $u(x, 0) = x$, hence $c_1 x + c_3 = x$, so $c_1 = 1$ and $c_3 = 0$

Therefore

$$u(x, y) = x + y$$