Numerical analysis 2 final exam correction 2025/2026

Solution of exercise 1: (12 pts)

1. $A = \begin{pmatrix} 6 & 1 & 1 \\ 2 & 4 & 0 \\ 1 & 2 & 6 \end{pmatrix}$; $b = \begin{pmatrix} 12 \\ 0 \\ 6 \end{pmatrix}$ ……………………………………..(1pts)

2. Gauss Elimination ……………………………………………………………(1pts)

$$\left(\begin{array}{ccc|c} 6 & 1 & 1 & 12 \\ 2 & 4 & 0 & 0 \\ 1 & 2 & 6 & 6 \end{array}\right) \to \left(\begin{array}{ccc|c} 6 & 1 & 1 & 12 \\ 0 & \frac{11}{3} & \frac{-1}{3} & -4 \\ 0 & \frac{11}{6} & \frac{35}{6} & 4 \end{array}\right) \to \left(\begin{array}{ccc|c} 6 & 1 & 1 & 12 \\ 0 & \frac{11}{3} & \frac{-1}{3} & -4 \\ 0 & 0 & 6 & 6 \end{array}\right)$$

$x_1 = 2,\quad x_2 = -1,\qquad x_3 = 1$……………………………………..…(1.5pts)

3. $Det(A) = \begin{vmatrix} 6 & 1 & 1 \\ 0 & \frac{11}{3} & \frac{-1}{3} \\ 0 & 0 & 6 \end{vmatrix} = 6 \times \frac{11}{3} \times 6 = 132$ …………………………………(1pts)

4. A is strictly diagonally dominant: $|a_{1,1}| = 6 > 1 + 1$; $|a_{2,2}| = 4 > 2 + 0$;$|a_{3,3}| = 6 > 1 + 2$ …………………………………………… ……………………………………….0.5pts)

5. $D = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}$; $L = \begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ -1 & -2 & 0 \end{pmatrix}$; $U = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ .... (0.5pts)

matrix of the Gauss-Seidel

$B_{GS} = (D - L)^{-1}.U = \begin{pmatrix} 0 & -\frac{1}{6} & -\frac{1}{6} \\ 0 & \frac{1}{12} & \frac{1}{12} \\ 0 & 0 & 0 \end{pmatrix}$ ……………………..……(0.5pts)

Eigenvalues of B_{GS}: $|B_{GS} - \lambda I| = \begin{vmatrix} \lambda & -\frac{1}{6} & -\frac{1}{6} \\ 0 & \lambda - \frac{1}{12} & \frac{1}{12} \\ 0 & 0 & \lambda \end{vmatrix} = \lambda^3 - \frac{1}{12}\lambda^2, \lambda = 0, \frac{1}{12}$ ……(2pts)

$\rho(B_{GS}) = \max\limits_{i=1,2,3} |\lambda_i| = \frac{1}{12} < 1$ Gauss-Seidel method is converge........(1pts)

6. $\begin{cases} x_1^{(k+1)} = 1/6(12 - x_2^{(k)} - x_3^{(k)}) \\ x_2^{(k+1)} = -1/2x_1^{(k+1)} \\ x_3^{(k+1)} = 1/6(6 - x_1^{(k+1)} - 2x_2^{(k+1)}) \end{cases}$ $\quad \text{Gauss} - \text{Seidel}$ (0.5 pts)

$\begin{cases} x_1^{(k+1)} = 1/6(12 - x_2^{(k)} - x_3^{(k)}) \\ x_2^{(k+1)} = -1/2x_1^{(k)} \\ x_3^{(k+1)} = 1/6(6 - x_1^{(k)} - 2x_2^{(k)}) \end{cases}$ Jacobi (0.5pts)

$x_1^{(1)} = 1.3333 \qquad x_1^{(2)} = 1.9444 \qquad x_1^{(3)} = 1.9259$

$x_2^{(1)} = -0.6667 \qquad x_2^{(2)} = -0.9722 \qquad x_2^{(3)} = -0.9977$ ....(1pts)

$x_3^{(1)} = 1 \qquad x_3^{(2)} = 1 \qquad x_3^{(3)} = 1$

$x_1^{(1)} = 2 \qquad x_1^{(2)} = 1.8333 \qquad x_1^{(3)} = 2.0556$

$x_2^{(1)} = 0 \qquad x_2^{(2)} = -1 \qquad x_2^{(3)} = -0.9167$ ....(1pts)

$x_3^{(1)} = 1 \qquad x_3^{(2)} = 0.6667 \qquad x_3^{(3)} = 1.0278$

Solution of exercise 2:(8pts)

$$\begin{cases} y'(x) = -y(x) + x, & x \in [0,1] \\ y(0) = 1 \end{cases}$$

1.• $f(x,y) = -y + x$ is continuous for $0 \leq x \leq 1$ and $-\infty < y < \infty$. ………(0.5pts)

• $|f(x,y_2) - f(x,y_1)| = |y_2 - y_1|$, f satisfies a Lipschitz condition

in the variable *y* with Lipschitz constant *L=1* ……………………….………….(0.5pts)

So, by the theorem, a unique solution exists to this initial-value problem.

3. Verify that $y = x - 1 + 2e^{-x}$ is the exact solution……………………………….(0.5)

2. The explicit Euler method *h=0.1*, $x_0 = 0, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3, y_0 = 1$ ...(1.5pts)

$$y_{n+1} = y_n + h\, f(x_n;\, y_n)$$

$$y_1 = y_0 + h\, f(x_0;\, y_0) = 0.9000$$

$$y_2 = y_1 + h\, f(x_1;\, y_1) = 0.8200$$

$$y_3 = y_2 + h\, f(x_2;\, y_2) = 0.7580$$

3. $E = |y(0.3) - y_3| = |0.7816 - 0.7580| = 0.0236$ ....(1pts)

4. approximation of $y'(0.3)$: $y'(0.3) = -y(0.3) + 0.3 = -0.4580$.......................(1pts)

$$E = |y'(0.3) - y'_{app}| = |-0.4816 + 0.4580| = 0.0236$$

5.The Runge-Kutta method of order 4, *h=0.2*...(1.5pts)

$$k_1 = h\, f(x_0;\, y_0) = -1$$

$$k_2 = h\, f\left(x_0 + \frac{h}{2};\, y_0 + \frac{k_1}{2}\right) = -0.8$$

$$k_3 = h\, f\left(x_0 + \frac{h}{2};\, y_0 + \frac{k_2}{2}\right) = -0.82$$

$$k_4 = h\, f(x_0 + h;\, y_0 + k_3) = -0.636$$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.837467$$

6.$E = |y(0.2) - y_1| = 5.16 \times 10^{-6}$ ....(0.5pts)

7.The error estimate for each method ....(1pts)

Explicit Euler method: $|y(x_n) - y_n| \leq ch$

The Runge-Kutta method of order 4: $|y(x_n) - y_n| \leq ch^4$