Numerical analysis 2 final exam correction 2024/2025

Solution of exercise 1: (12 pts)

1. $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}; \; b = \begin{pmatrix} 5 \\ -5 \\ 3 \end{pmatrix}$ ……………………………………………..(1pts)

2. Gauss Elimination …………………………………………………………………(1pts)

$$\left(\begin{array}{ccc|c} 2 & -1 & 0 & 5 \\ -1 & 2 & -1 & -5 \\ 0 & -1 & 2 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & -1 & 0 & 5 \\ 0 & \frac{3}{2} & -1 & -\frac{5}{2} \\ 0 & -1 & 2 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & -1 & 0 & 5 \\ 0 & \frac{3}{2} & -1 & -\frac{5}{2} \\ 0 & 0 & \frac{4}{3} & \frac{4}{3} \end{array}\right)$$

$x_1 = 2, \quad x_2 = -1, \quad x_3 = 1$……………………………………………..…(1.5pts)

3. $Det(A) = \begin{vmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{vmatrix} = 2 \times \frac{3}{2} \times \frac{4}{3} = 4$ ……………………………………(1pts)

4. A is symmetric $a_{i,j} = a_{j,i}$ ……………………………………………………………(0.5pts)

A is positive definite $x^t A x > 0$ for all $x \in \mathbb{R}^3$; $x \neq 0$

$$x^t A x = 2{x_1}^2 - 2x_1x_2 + 2{x_2}^2 - 2x_2x_3 + 2{x_3}^2$$
$$= {x_1}^2 + {x_3}^2 + (x_2 - x_1)^2 + (x_3 - x_2)^2 > 0$$ ……..….(1pts)

5. $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}; \; L = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \; U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$……………….(0.5pts)

matrix of the Jacobi $B_J = D^{-1}.(L+U) = \begin{pmatrix} 0 & 0.5 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0 \end{pmatrix}$……………(0.5pts)

Eigenvalues of B_J: $|B_J - \lambda I| = \begin{vmatrix} \lambda & 0.5 & 0 \\ 0.5 & \lambda & 0.5 \\ 0 & 0.5 & \lambda \end{vmatrix} = -\lambda^3 + 2\lambda, \lambda = 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$ …..(2pts)

$\rho(B_J) = \max_{i=1,2,3} |\lambda_i| = \frac{1}{\sqrt{2}} \approx 0.7071 < 1$ is Jacobi's method converge …………………(1pts)

6. $\begin{cases} {x_1}^{(k+1)} = 0.5{x_2}^{(k)} + 2.5 \\ {x_2}^{(k+1)} = 0.5{x_2}^{(k)} + 0.5{x_3}^{(k)} - 2.5 \\ {x_3}^{(k+1)} = 0.5{x_2}^{(k)} + 1.5 \end{cases}$ ……………………………………………(0.5pts)

${x_1}^{(1)} = 2.5 \qquad {x_1}^{(2)} = 1.25 \qquad {x_1}^{(3)} = 2.25$

${x_2}^{(1)} = -2.5 \qquad {x_2}^{(2)} = -0.5 \qquad {x_2}^{(3)} = -1.75$ ……………………………(1.5pts)

${x_3}^{(1)} = 1.5 \qquad {x_3}^{(2)} = 0.25 \qquad {x_3}^{(3)} = 1.25$

Solution of exercise 2:(8pts)

$$\begin{cases} y'(x) = -2x\,y(x), & x \in [0,1] \\ y(0) = 1 \end{cases}$$

1.•$f(x,y) = -2xy$ is continuous for $0 \leq x \leq 1$ and $-\infty < y < \infty$. …………(0.5pts)

•$|f(x,y_2) - f(x,y_1)| = 2|x||y_2 - y_1| < 2|y_2 - y_1|$, f satisfies a Lipschitz condition

in the variable *y* with Lipschitz constant *L=2* ……………………..…………(0.5pts)

So, by the theorem, a unique solution exists to this initial-value problem.

2. The explicit Euler method *h=0.1*, $x_0 = 0, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3$, $y_0 = 1$ ....(2pts)

$y_{n+1} = y_n + h\,f(x_n;\, y_n)$

$y_1 = y_0 + h\,f(x_0;\, y_0) = 1$

$y_2 = y_1 + h\,f(x_1;\, y_1) = 0.9800$

$y_3 = y_2 + h\,f(x_2;\, y_2) = 0.9408$

3. approximation of $y'(0.3)$: $y'(0.3) = -2(0.3)(0.9408) = -0.5645$ ....(1pts)
4. $E = |y(0.3) - y_3| = 0.0269$ ....(1pts)
5.The Runge-Kutta method of order 4, *h=0.2* ....(1.5pts)

$k_1 = h\,f(x_0;\, y_0) = 0$

$k_2 = h\,f\left(x_0 + \frac{h}{2};\, y_0 + \frac{k_1}{2}\right) = -0.0400$

$k_3 = h\,f\left(x_0 + \frac{h}{2};\, y_0 + \frac{k_2}{2}\right) = -0.0392$

$k_4 = h\,f(x_0 + h;\, y_0 + k_3) = -0.0769$

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.9608$

6.$E = |y(0.2) - y_1| = 1.0561.10^{-5}$ ....(0.5pts)

7.The error estimate for each method ....(1pts)

Explicit Euler method: $|y(x_n) - y_n| \leq ch$

The Runge-Kutta method of order 4: $|y(x_n) - y_n| \leq ch^4$