Solution Numerical Analysis Exam

Exercise 01 Solution

Consider the system:

$$(*) \begin{cases} 4x_1 + 2x_2 + x_3 = 6 \\ 2x_1 + 5x_2 + x_3 = 10 \\ x_1 + x_2 + 3x_3 = -3 \end{cases}$$

1. **Matrix Form $Ax = b$:**

$$A = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 5 & 1 \\ 1 & 1 & 3 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad b = \begin{bmatrix} 6 \\ 10 \\ -3 \end{bmatrix}$$

So, the system becomes:

$$Ax = b$$

2. **LU Factorization Existence:**

A matrix admits LU factorization without pivoting if all leading principal minors are non-zero:

$$\det(A_1) = 4 \neq 0$$
$$\det(A_2) = \begin{vmatrix} 4 & 2 \\ 2 & 5 \end{vmatrix} = 20 - 4 = 16 \neq 0$$
$$\det(A) = \begin{vmatrix} 4 & 2 & 1 \\ 2 & 5 & 1 \\ 1 & 1 & 3 \end{vmatrix} = 43 \neq 0$$

Conclusion: A admits a unique LU factorization.

3. **Symmetric Positive Definiteness:**

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- A is strictly diagonally dominant:

$$|a_{11}| = 4 > |2| + |1| = 3$$
$$|a_{22}| = 5 > |2| + |1| = 3$$
$$|a_{33}| = 3 > |1| + |1| = 2$$

- All diagonal elements are strictly positive: $a_{ii} > 0$ for all i.

Conclusion: Since A is symmetric and strictly diagonally dominant with positive diagonal entries, it is symmetric positive definite.

4. **Cholesky Factorization $A = LL^T$:**

Assume:

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

Solving $LL^T = A$, we get:

$$\begin{aligned}
l_{11} &= \sqrt{4} = 2 \\
l_{21} &= \frac{2}{2} = 1 \\
l_{31} &= \frac{1}{2} = 0.5 \\
l_{22} &= \sqrt{5 - 1^2} = \sqrt{4} = 2 \\
l_{32} &= \frac{1 - (\frac{1}{2})(1)}{2} = \frac{1}{4} \\
l_{33} &= \sqrt{3 - (\frac{1}{2})^2 - (\frac{1}{4})^2} = \frac{\sqrt{43}}{4}
\end{aligned}$$

Therefore,

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{\sqrt{43}}{4} \end{bmatrix}$$

5. **Solve the System Using Cholesky:**

   Let $LL^T x = b$. Let $y = L^T x$, then solve:

$$Ly = b \Rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{\sqrt{43}}{4} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ -3 \end{bmatrix}$$

$$\begin{aligned}
2y_1 = 6 &\Rightarrow y_1 = 3 \\
y_1 + 2y_2 = 10 &\Rightarrow y_2 = \frac{7}{2} \\
\frac{1}{2}(3) + \frac{1}{4}(\frac{7}{2}) + \frac{\sqrt{43}}{4} y_3 = -3 &\Rightarrow y_3 = -\frac{\sqrt{43}}{2}
\end{aligned}$$

   Now solve $L^T x = y$:

$$L^T = \begin{bmatrix} 2 & 1 & \frac{1}{2} \\ 0 & 2 & \frac{1}{4} \\ 0 & 0 & \frac{\sqrt{43}}{4} \end{bmatrix}$$

$$\frac{\sqrt{43}}{4} x_3 = -\frac{\sqrt{43}}{2} \Rightarrow x_3 = -22x_2 + \frac{1}{4}(-2) \quad = \frac{7}{2} \Rightarrow x_2 = 2$$

$$2x_1 + 2 + (-1) = 3 \Rightarrow x_1 = 1$$

$$\boxed{x = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}}$$

6. **Convergence of Jacobi and Gauss-Seidel:**

   Since A is symmetric positive definite, both Jacobi and Gauss-Seidel methods converge for any $x^{(0)} \in \mathbb{R}^3$.

7. **Jacobi and Gauss-Seidel Iteration Systems:**

   From the original system:

$$\begin{cases} x_1 = \frac{1}{4}(6 - 2x_2 - x_3) \\ x_2 = \frac{1}{5}(10 - 2x_1 - x_3) \\ x_3 = \frac{1}{3}(-3 - x_1 - x_2) \end{cases}$$

Jacobi Iteration:

$$\begin{cases} x_1^{(k+1)} = \frac{1}{4}(6 - 2x_2^{(k)} - x_3^{(k)}) \\ x_2^{(k+1)} = \frac{1}{5}(10 - 2x_1^{(k)} - x_3^{(k)}) \\ x_3^{(k+1)} = \frac{1}{3}(-3 - x_1^{(k)} - x_2^{(k)}) \end{cases}$$

Gauss-Seidel Iteration:

$$\begin{cases} x_1^{(k+1)} = \frac{1}{4}(6 - 2x_2^{(k)} - x_3^{(k)}) \\ x_2^{(k+1)} = \frac{1}{5}(10 - 2x_1^{(k+1)} - x_3^{(k)}) \\ x_3^{(k+1)} = \frac{1}{3}(-3 - x_1^{(k+1)} - x_2^{(k+1)}) \end{cases}$$

Exercise 02 Solution

Given the Cauchy problem:

$$y'(t) = t + 1 - y(t), \quad y(0) = 1$$

1. Exact solution:

$$y(t) = e^{-t} + t \quad \text{(can be verified by substitution)}$$

2. Euler method with $h = 0.1$, from $t = 0$ to $t = 0.3$:

$$\begin{aligned} y_0 &= 1 \\ y_1 &= y_0 + h(f(0, y_0)) = 1 + 0.1(0 + 1 - 1) = 1 \\ y_2 &= y_1 + 0.1(0.1 + 1 - 1) = 1 + 0.1(0.1) = 1.01 \\ y_3 &= y_2 + 0.1(0.2 + 1 - 1.01) = 1.01 + 0.1(0.19) = 1.029 \end{aligned}$$

So, $y(0.3) \approx 1.029$

3. Approximate $y'(0.3) \approx f(0.3, 1.029) = 0.3 + 1 - 1.029 = 0.271$

4. Exact $y(0.3) = e^{-0.3} + 0.3 \approx 0.7408 + 0.3 = 1.0408$ Difference: $|1.029 - 1.0408| \approx 0.0118$

5. Runge-Kutta order 4 with $h = 0.2$:

$k_1 = f(0, 1) = 0+1-1 = 0$

$k_2 = f(0.1, 1) = 0.1+1-1 = 0.1$

$k_3 = f(0.1, 1.005) = 0.1+1-1.005 = 0.095$

$k_4 = f(0.2, 1.019) = 0.2+1-1.019 = 0.181$

$$y(0.2) \approx 1 + \frac{0.2}{6}(0 + 2(0.1) + 2(0.095) + 0.181) \approx 1 + 0.0337 = 1.0337$$

6. Exact $y(0.2) = e^{-0.2} + 0.2 \approx 0.8187 + 0.2 = 1.0187$ Difference: $|1.0337 - 1.0187| = 0.015$