Chapter 03

Numerical Solutions of First-Order ODEs

Cauchy Problem

We limit our course to the case of ordinary differential equations of the form :

$$\begin{cases} \text{Find the function } y(x) \text{ such that :} \\ y'(x) = f(x, y(x)) \quad \forall x \in [a, b] \\ y(x_0) = y_0 \end{cases}$$

Definition A function $f(x, y)$ is said to satisfy a *Lipschitz condition* in the variable y on a set $D \subset \mathbb{R}^2$ if a constant $L > 0$ exists such that

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|,$$

whenever (x, y_1) and (x, y_2) are in D. The constant L is called a *Lipschitz constant* for f.

Example Show that $f(x, y) = x|y|$ satisfies a Lipschitz condition on the domain

$$D = \{(x, y) \mid 1 \leq x \leq 2, \ -3 \leq y \leq 4\}.$$

Solution. For each pair of points (x, y_1) and (x, y_2) in D, we have

$$\begin{aligned} |f(x, y_1) - f(x, y_2)| &= |x|y_1| - x|y_2|| \\ &= |x| \cdot ||y_1| - |y_2|| \\ &= |x| \cdot ||y_1| - |y_2||. \end{aligned}$$

Using the inequality $||a| - |b|| \leq |a - b|$, we get

$$|f(x, y_1) - f(x, y_2)| \leq |x| \cdot |y_1 - y_2|.$$

Now, since $x \in [1, 2]$, it follows that $|x| \leq 2$. Therefore,

$$|f(x, y_1) - f(x, y_2)| \leq 2|y_1 - y_2|.$$

Thus, the function $f(x, y) = |y|$ satisfies a Lipschitz condition in y on the domain D with Lipschitz constant $L = 2$.

Theorem. Suppose $f(x, y)$ is defined on a convex set $D \subset \mathbb{R}^2$. If a constant $L > 0$ exists such that

$$\left| \frac{\partial f}{\partial y}(x, y) \right| \leq L, \quad \text{for all } (x, y) \in D,$$

then f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant L.

Theorem. Suppose that $D = \{(x, y) \mid a \leq x \leq b, -\infty < y < \infty\}$, and that $f(x, y)$ is continuous on D. If f satisfies a Lipschitz condition on D in the variable y, then the initial-value problem

$$y'(x) = f(x, y), \quad a \leq x \leq b, \quad y(a) = \alpha,$$

has a unique solution $y(x)$ for $a \leq x \leq b$.

Example. Use Theorem 5.4 to show that there is a unique solution to the initial-value problem

$$y' = 1 + x\sin(xy), \quad 0 \le x \le 2, \quad y(0) = 0.$$

Solution. Holding x constant and applying the Mean Value Theorem to the function

$$f(x, y) = 1 + x\sin(xy),$$

we find that when $y_1 < y_2$, there exists a number $\xi \in (y_1, y_2)$ such that

$$\frac{f(x, y_2) - f(x, y_1)}{y_2 - y_1} = \frac{\partial f}{\partial y}(x, \xi) = x^2\cos(\xi x).$$

Thus,

$$|f(x, y_2) - f(x, y_1)| = |y_2 - y_1| \cdot |x^2\cos(\xi x)| \le 4|y_2 - y_1|,$$

since $x \in [0, 2]$ implies $x^2 \le 4$, and $|\cos(\xi x)| \le 1$. Therefore, f satisfies a Lipschitz condition in the variable y with Lipschitz constant $L = 4$.

Additionally, $f(x, y)$ is continuous for $0 \le x \le 2$ and $-\infty < y < \infty$, so by the theorem, a unique solution exists to this initial-value problem.

Euler's Method

Euler's method is a numerical technique used to approximate the solution of an ordinary differential equation (ODE) of the form :

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0,$$

where $y(x)$ is the unknown function, $f(x, y)$ is a given function, and $y(x_0) = y_0$ is the initial condition. The method approximates the solution by discretizing the interval $[x_0, x_{\text{end}}]$ into small steps of size h and iteratively computing the solution at each step.

explicit Euler Method

Given a step size h, the method computes approximate values y_n of the solution at points $x_n = x_0 + nh$, for $n = 0, 1, 2, \ldots$, using the iterative formula :

$$y_{n+1} = y_n + h \cdot f(x_n, y_n),$$

where :
- $x_n = x_0 + nh$,
- $y_n \approx y(x_n)$,
- h is the step size.

The process starts with the initial condition $y_0 = y(x_0)$ and continues until the desired endpoint is reached.

Implicit Euler Method

Given :

$$y_0 \text{ given}, \quad x_{n+1} = x_n + h$$

The implicit Euler formula is :

$$y_{n+1} = y_n + h \cdot f(x_{n+1}, y_{n+1})$$

Example

Consider the initial value problem (IVP) :

$$\begin{cases} y' = -y + x + 1, \\ y(0) = 1, \end{cases}$$

with the exact solution :

$$y(x) = x + e^{-x}.$$

We solve the IVP using :
– Euler's Explicit Method,
– Euler's Implicit Method,
for step size $h = 0.2$ and compute the approximate solution at $x = 0.2, 0.4, 0.6, 0.8, 1.0$.

1. Euler's Explicit Method

The formula for the explicit method is :

$$y_{n+1} = y_n + hf(x_n, y_n),$$

where $f(x, y) = -y + x + 1$.

2. Euler's Implicit Method

The formula for the implicit method is :

$$y_{n+1} = y_n + hf(x_{n+1}, y_{n+1}),$$

which gives :

$$y_{n+1} = y_n + h(-y_{n+1} + x_{n+1} + 1).$$

Solving for y_{n+1} :

$$(1 + h)y_{n+1} = y_n + h(x_{n+1} + 1),$$

$$y_{n+1} = \frac{y_n + h(x_{n+1} + 1)}{1 + h}.$$

3. Comparison Table

x	Exact $y(x)$	Explicit y_E	Error (Explicit)	Implicit y_I	Error (Implicit)
0.0	1.00000	1.00000	0.00000	1.00000	0.00000
0.2	1.01873	1.00000	0.01873	1.03333	0.01460
0.4	1.07032	1.04000	0.03032	1.09444	0.02412
0.6	1.14881	1.11200	0.03681	1.17870	0.02989
0.8	1.24933	1.20960	0.03973	1.28225	0.03292
1.0	1.36788	1.32768	0.04020	1.40188	0.03400

Both the explicit and implicit Euler methods provide reasonable approximations of the exact solution. The implicit method offers slightly better stability, but in this case, the error increases similarly with both methods due to the relatively large step size. **Theorem.** Suppose that the function $f(x,y)$ is continuous in both variables and Lipschitz continuous in y uniformly in x, and that $y \in \mathcal{C}^2[a,b]$. Let $M_2 = \max_{x\in[a,b]} |y''(x)|$. Then we have the estimate

$$|y_n - y(x_n)| \leq (e^{L(b-a)} - 1)\frac{M_2}{2L}h$$

Second-Order Taylor Method

The Taylor expansion provides a first generalization of Euler's method. The idea is to replace, over the interval $[x_n, x_{n+1}]$, the curve $y(x)$, which is the solution of the differential equation, not by a straight line, but by a parabola.

Algorithm : Second-Order Taylor Method

1. Given a step size h, an initial condition (x_0, y_0), and a maximum number of iterations N.
2. For $0 \leq n \leq N$:
   - $$y_{n+1} = y_n + hf(t_n, y_n) + \frac{h^2}{2}\left(\frac{\partial f(x_n,y_n)}{\partial x} + \frac{\partial f(x_n,y_n)}{\partial y}f(x_n,y_n)\right)$$
   - $x_{n+1} = x_n + h$

   3. Stop.

   Remark
   We observe that in this algorithm as well, errors propagate from one iteration to the next.

   Theorem
   Suppose that $f(x,y) \in \mathscr{C}^2([a,b]\times\mathbb{R})$, f is L_0-Lipschitz continuous, and

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f$$

is L_1-Lipschitz continuous with respect to y, uniformly with respect to x. Let

$$M_3 = \max_{x\in[a,b]} |y^{(3)}(x)|.$$

Then the following error estimate holds :

$$|y_n - y(x_n)| \le \left(e^{\left(L_0 + \frac{b-a}{2}L_1\right)(b-a)} - 1\right) \frac{M_3}{6L_0} h^2$$

Example
We are given :

$$y' = xy + 1, \quad y(0) = 1, \quad h = 0.1$$

Let $f(x, y) = xy + 1$. Then :

$$f'(x, y) = \frac{d}{dx}(xy + 1) = y + x \cdot (xy + 1) = y(1 + x^2) + x$$

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Step 1 : At $x_0 = 0,\ y_0 = 1$:

$$f(x_0, y_0) = 0 \cdot 1 + 1 = 1$$

$$f'(x_0, y_0) = 1(1 + 0^2) + 0 = 1$$

$$y_1 = y_0 + hf(x_0, y_0) + \frac{h^2}{2} f'(x_0, y_0) = 1 + 0.1 \cdot 1 + \frac{0.1^2}{2} \cdot 1 = 1 + 0.1 + 0.005 = 1.105$$

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Step 2 : At $x_1 = 0.1,\ y_1 = 1.105$:

$$f(x_1, y_1) = 0.1 \cdot 1.105 + 1 = 1.1105$$

$$f'(x_1, y_1) = 1.105(1 + 0.1^2) + 0.1 = 1.105(1.01) + 0.1 = 1.21605$$

$$y_2 = y_1 + hf(x_1, y_1) + \frac{h^2}{2} f'(x_1, y_1) = 1.105 + 0.1 \cdot 1.1105 + \frac{0.1^2}{2} \cdot 1.21605$$

$$= 1.105 + 0.11105 + 0.00608025 \approx \boxed{1.22213}$$

Runge-Kutta Methods

Second-Order Runge-Kutta (RK2)

The second-order Runge-Kutta method, also known as the Heun method or Modified Euler Method, is given by :

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf(x_n + h, y_n + k_1)$$

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

Example 1 : Solving $\frac{dy}{dx} = -2xy,\ y(0) = 1$ with RK2

Let's choose a step size $h = 0.1$.
- Initial condition : $x_0 = 0$, $y_0 = 1$ - Step size : $h = 0.1$
The function is $f(x, y) = -2xy$. We can now apply the RK2 method :
1. Calculate k_1 :

$$k_1 = h \cdot f(x_0, y_0) = 0.1 \cdot (-2 \cdot 0 \cdot 1) = 0$$

2. Calculate k_2 :

$$k_2 = h \cdot f(x_0 + h, y_0 + k_1) = 0.1 \cdot (-2 \cdot 0.1 \cdot 1) = -0.02$$

3. Update y_1 :

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2) = 1 + \frac{1}{2}(0 - 0.02) = 1 - 0.01 = 0.99$$

Thus, the approximate value of y at $x = 0.1$ is $y_1 = 0.99$.

Fourth-Order Runge-Kutta (RK4)

The fourth-order Runge-Kutta method is more accurate and is given by the following steps :

$$k_1 = hf(x_n, y_n)$$
$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$
$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$
$$k_4 = hf(x_n + h, y_n + k_3)$$
$$y_{n+1} = y_n + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

Example 2 : Solving $\frac{dy}{dx} = -2xy,\ y(0) = 1$ with RK4

using a step size $h = 0.1$.
- Initial condition : $x_0 = 0$, $y_0 = 1$ - Step size : $h = 0.1$
The function is $f(x, y) = -2xy$. Now, let's apply the RK4 method :
1. Calculate k_1 :

$$k_1 = h \cdot f(x_0, y_0) = 0.1 \cdot (-2 \cdot 0 \cdot 1) = 0$$

2. Calculate k_2 :

$$k_2 = h \cdot f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1 \cdot (-2 \cdot 0.05 \cdot 1) = -0.01$$

3. Calculate k_3 :

$$k_3 = h \cdot f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1 \cdot (-2 \cdot 0.05 \cdot 0.995) = -0.00100$$

4. Calculate k_4 :

$$k_4 = h \cdot f(x_0 + h, y_0 + k_3) = 0.1 \cdot (-2 \cdot 0.1 \cdot (1 - 0.00995)) = -0.0198$$

5. Update y_1 :

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1 + \frac{1}{6}(0 + 2(-0.01) + 2(-0.00100) + (-0.0198))$$

$$y_1 = 1 + \frac{1}{6}(-0.02 - 0.0199 - 0.0199) = 1 - 0.00995 \approx 0.99005$$

Thus, the approximate value of y at $x = 0.1$ is $y_1 \approx 0.99005$.

Theorem (Convergence of Numerical Methods)

For a Lipschitz continuous function f, the following error bounds hold for different numerical methods :

$$\begin{aligned}
\text{Forward/Backward Euler} : \|y(x_n) - y_n\| &\leq Ch, \\
\text{RK2 (Midpoint method)} : \|y(x_n) - y_n\| &\leq Ch^2, \\
\text{RK4 (Classical Runge-Kutta)} : \|y(x_n) - y_n\| &\leq Ch^4,
\end{aligned}$$

where :

- $y(x_n)$ is the exact solution at time x_n,
- y_n is the numerical approximation,
- h is the step size,
- C is a constant depending on the Lipschitz constant of f and the smoothness of the exact solution.